# DIRECTORATE OF DISTANCE EDUCATION

# UNIVERSITY OF NORTH BENGAL

**MASTERS OF SCIENCE-MATHEMATICS**

**SEMESTER -I**

# DIFFERENTIAL GEOMETRY

**DEMATH1ELEC4**

# BLOCK-2

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# FOREWORD

The Self-Learning Material (SLM) is written with the aim of providing simple and organized study content to all the learners. The SLMs are prepared on the framework of being mutually cohesive, internally consistent and structured as per the university's syllabi. It is a humble attempt to give glimpses of the various approaches and dimensions to the topic of study and to kindle the learner's interest to the subject

We have tried to put together information from various sources into this book that has been written in an engaging style with interesting and relevant examples. It introduces you to the insights of subject concepts and theories and presents them in a way that is easy to understand and comprehend.

We always believe in continuous improvement and would periodically update the content in the very interest of the learners. It may be added that despite enormous efforts and coordination, there is every possibility for some omission or inadequacy in few areas or topics, which would definitely be rectified in future.

We hope you enjoy learning from this book and the experience truly enrich your learning and help you to advance in your career and future endeavours.

# DIFFERENTIAL GEOMETRY

## BLOCK - I



## BLOCK - II

# BLOCK-2 DIFFERENTIAL GEOMETRY

**Introduction to the Block**

In this block we will go through Geometry…

Unit VIII Deals with Geometry

Unit IX Deals with Linear Algebra

Unit X Deals with The Notion of Curve

Unit XI Deals with Plane Curves

Unit XII Deals with The Gauss—Bonnet Formula

Unit XIII Deals with The Principal Axis Theorem

Unit XIV Deals with ssssThe Frenet - Serret Formulas

# UNIT –8 : GEOMETRY

**STRUCTURE**



## 8.0 OBJECTIVES

After studying this unit , you should be able to:

- Understand about Geometry
- Euclidean Spaces
- Cartesian Coordinate Systems
- Linear Diffrential Equations

## 8.1 INTRODUCTION

Differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces Mathematical analysis of curves and surfaces had been developed to answer some of unanswered questions that appeared in calculus like the reasons for relationships between complex shapes and curves , series and analytic functions Geometry , Euclidean Spaces , Cartesian Coordinate Systems , Linear Diffrential Equations

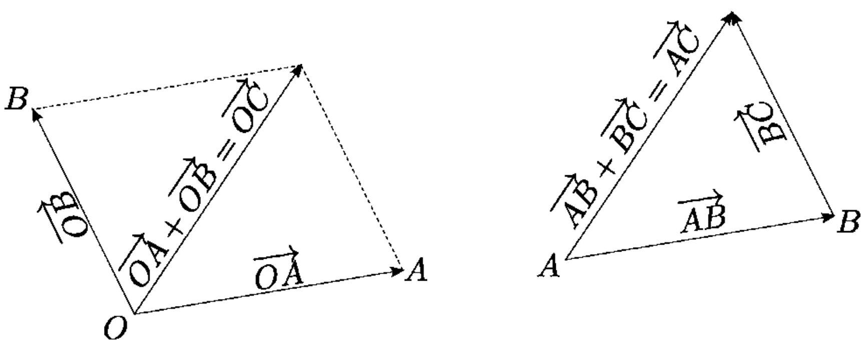
# 8.2 GEOMETRY

Affine Geometry… Affine Spaces

In traditional axiomatic treatment of Euclidean geometry , vectors are defined as equivalence classes of ordered pairs of points ( also called directed seg - ments ) . If X is the set of points of the space , then an ordered pair of points is simply an element of the Cartesian product X x X . If ( A , B ) G X x X is an ordered pair of points , then A is called the initial point , and B is called the endpoint of the ordered pair . The ordered pairs of points ( A , B ) and ( C , D ) are said to be equivalent if the midpoint of the segment [A , D] coincides with the midpoint of the segment [B , C] . It can be shown that this is indeed an equivalence relation . The equivalence classes are called ( free ) vectors . The set V of all vectors is equipped with a linear space structure . The sum of two vectors is constructed by the triangle or parallelogram rule , multiplication by scalars is defined in the usual way . The map $: X x X ^ V , which assigns to each ordered pair of points ( A , B ) the vector $ ( A , B ) = aB represented by it , i . e . , its equivalence class , satisfies the following properties .

( A1 ) For any A G X and v G V there is a unique B G X such that T ( A , B ) = v .

( A2 ) ( Triangle rule . ) For any three points A , B , C G X , we have T ( A , B ) + T ( B , C ) = T ( A , C ) .




The construction of the sum of two vectors by the triangle and the parallelogram rule .

The notion of an affine space generalizes this picture .

Definition . An affine space is a triple A = ( X , V , T ) , where X is a set , the elements of which are called points , V is a linear space , the elements of which are called vectors , and T: X x X ^ V is a map satisfying conditions ( A1 ) and ( A2 ) above . The dimension of the affine space A is the ( linear algebraic ) dimension of the linear space V .

When it leads to no confusion , T ( A , B ) is also denoted by aB .

Definition Let A = ( X , V , T ) and B = ( Y , W , T ) be two affine spaces . Then an affine transformation from A to B consists of a map T: X ^ Y and a linear map L: V ^ W such that

L ( T ( A , B ) ) = T ( T ( A ) , T ( B ) ) for all A , B G X .

Observe that since T is surjective , T determines L uniquely by

Affine spaces form a category in which the morphisms are the affine trans - formations . An affine transformation is an isomorphism if and only if T is a bijection .

Example . Let V be an arbitrary linear space , X = V , and define T: X x X ^ V by T ( p , q ) = q - p . Then AV = ( X , V , T ) is an affine space . Thus , every linear space carries an affine space structure . If V and W are linear spaces , then an affine transformation from AV to AW has the form T ( v ) = L ( v ) + wo , where L: V ^ W is a linear map , w0 G W is a fixed vector .

Linear spaces and affine spaces are very similar objects . The main difference is that in a linear space , we always have a distinguished point , the origin 0 , whereas in an affine space , none of the points of X is distinguished . This is essentially the only difference , because if we choose any of the points of X for the origin , we can turn X into a linear space isomorphic to V .

Indeed , choose a point O G X . By the first axiom ( A1 ) of an affine space , the map TO: X ^ V , TO ( P ) = T ( O , P ) is a bijection between X and V , and we have

T ( A , B ) = To ( B ) - To ( A )

by ( A2 ) . This identity means that TO together with the linear map L = idV is an isomorphism between the affine space A = ( X , V , T ) and the affine space Av .

The vector To ( P ) is called the position vector of A from the base point O . Identification of points of an affine space with vectors with the help of a fixed base point O is called vectorization of the affine space .

## Affine Subspaces

Definition . A nonempty subset Y C X of the point set of an affine space A = ( X , V , T ) is an affine subspace of A if the image T ( Y x Y ) = W of Y x Y under T is a linear subspace of V and ( Y , W , T|YxY ) is an affine space . W is called the direction space of the affine subspace , its elements are the direction vectors of Y , or the vectors parallel to Y . Since affine subspaces are affine spaces themselves , their dimension is properly defined .

The 0 - dimensional affine subspaces of an affine space A = ( X , V , T ) are the points of X . The 1 - dimensional affine subspaces are called straight lines . The 2 - dimensional affine subspaces are the ordinary planes of A . In gen - eral , k - dimensional affine subspaces of an affine space will be called shortly k - planes . The ( n — 1 ) - planes of an n - dimensional affine space are called hy - perplanes . The set of all k - dimensional affine subspaces of an affine space A is the affine Grassmann manifold AGrk ( A ) . At this moment we defined the affine Grassmann manifold just as a set , but this set can be endowed with a ( k + 1 ) ( n — k ) - dimensional manifold structure , which justifies its name . The following proposition gives a more explicit description of affine subspaces in vectorized affine spaces .

Proposition . A subset Y C V of the affine space Av is an affine subspace with direction space W < V if and only if Y is a translate of W .

Proof . If Y is an affine subspace with direction space W , then according to axiom ( A1 ) , for any a G Y , the map Y ^ W , b ^ b — a is a bijection between Y and W , which means that Y is the translate of W by the vector a .

Assume now that Y = Ta ( W ) is a translate of the linear subspace W of V , and show that Y is an affine subspace of AV with direction space W . Two typical elements of Y have the form yi = a + wi and y2 = a + w2 , where wi , w2 G W . Then

$ ( yi , y2 ) = y2 — yi = w2 — wi , which shows that T ( Y x Y ) = W .

To check axiom ( A1 ) for ( Y , W , $|YxY ) , observe that for y0 = a + w0 G Y , the map Y ^ W , y ^ T ( y0 , y ) = y — y0 is a translation by — y0 restricted onto Y . Since T - y0 = T - W0 o T - a and T - a maps Y onto W bijectively , and T - W0 maps W onto itself bijectively , T - y0 maps Y onto W bijectively .

Since the triangle rule ( A2 ) is fulfilled for any triple in X , it is true for any three points in Y as well .

Affine and 0 - weight Linear Combinations

When we vectorize an affine space , points are identified with vectors and , therefore , we can take linear combinations of points . The result can be con - sidered both a point and a vector . However , the resulting point , or vector may depend on the choice of the base point of the vectorization . From the viewpoint of affine geometry , linear combinations the result of which does not depend on the choice of the vectorization are of special importance .

Let us compare vectorizations with base points O and O' . If T ( O , O' ) = a , and a point P corresponds to the vectors ( P ) = P and ( P ) = p' under the vectorizations with these base points , then we have p = p' + a by the triangle rule ( A2 ) .

If the position vectors of the points Pi , . . . , Pk G X from the base point O are pi , . . . , pk , then the linear combination Aipi + • • • + Akpk as a vector in V does not depend on the vectorization if and only if

Aipi + • • • + Afc pfc = Ai ( pi — a ) + • • • + Ak ( pk — a )

for any a G V , or , equivalently , if Ai + • • • + Ak =0 .

Definition . A linear combination Aipi + • • • + Akpk is a 0 - weight linear combination if the sum Ai + • • • + Ak of the coefficients is equal to 0 .

According to the previous computation , we obtain the following statement .

Proposition . For 0 - weight linear combinations the vector Aipi + • • • + AkPk € V does not depend on the vectorization , thus it can be denoted by AiPi + • • • + AkPk € V as well .

We get a different condition on the coefficients if we want that the point with position vector A1p1 + • • • + Akpk with base point O be the same as the point with position vector A1p'i + • • • + Akp'k with base point O' . For this we need that the equation

A1 p1 + • • • + Akpk — a = A1 ( p1 — a ) + • • • + Ak ( pk — a )

should hold for any choice of a . This condition is clearly equivalent to the condition that A1 + • • • + Ak = 1 .

Definition . A linear combination A1p1 + • • • + Akpk is an affine com - bination if the sum A1 + • • • + Ak of the coefficients is equal to 1 .
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The importance of affine combinations is summarized in the following proposition .

Proposition . The point represented by an affine combination Aipi + • • • + A k pk of the position vectors of some points Pi , . . . , Pk does not depend on the choice of the base point . This way , it makes sense to denote it by

AiPi + • • • + AkPk € X .

Affine subspaces can be characterized in terms of affine combinations .

Proposition . A subset Y of an affine space is an affine subspace if and only if it is nonempty and it contains a , ll affine combinations of its points .

Proof . Identify the space with AV by vectorization .

If Y is an affine subspace , then it is a translation TaW of its direction space . Since 0 € W , a € Y is not empty . Furthermore , if yj = a + Wj € Y , ( i = 1 , . . . , k ) are some points , then an affine combination of the has the form

Aiyi + • • • + Akyk = ( Ai + • • • + Ak ) a + ( Aiwi + • • • + Ak wk ) = a + ( Aiwi + + A k wk ) .

Since W is a linear space , w = Aiwi + • • • + Akwk € W , therefore ,

Aiyi + Ak yk = a + w € Y .

To show the other direction , assume now that Y is a nonempty subset of V containing all affine combinations of its points .

Consider the set Wof all 0 - weight linear combinations of elements of Y . Since the sum of two 0 - weight linear combinations and any multiple of a 0 - weight linear combination are 0 - weight linear combinations , W is a linear subspace of V .

Choose a vector a G Y , and let us show that Y = Ta ( W ) . Adding a = Pa to a 0 - weight linear combination of elements of Y we obtain an affine combination of some elements of Y , which belongs to Y by our assumptions on Y . Thus , Y D Ta ( W ) . On the other hand , every y element of Y can be written as y = a + ( y — a ) G Ta ( W ) , which completes the proof .

As a byproduct of the proof we obtain

Corollary . The direction space of an affine subspace is the set of all 0 - weight linear combinations of its elements .

Corollary . The intersection of affine subspaces of an affine space is either empty or it is also an affine subspace .

Corollary . For a nonempty subset S C X there is a smallest affine subspace among the affine subspaces containing S .

Definition . The smallest affine subspace containing the nonempty subset S is called the affine subspace spanned by S and it is denoted by aff [S] .

Proposition . The affine subspace spanned by the nonempty subset S consists of all affine combinations of the elements of S .

Proof . Denote by YS the set of all affine combinations of elements of S . By Proposition , aff[S] D YS . Since for any s G S , 1 • s is an affine combination of s , therefore S C YS . It is easy to check that affine combinations of affine combinations of elements of S is again an affine combination of elements of S , so YS is an affine subspace containing S . However , among such affine subspaces aff[S] is the smallest one , so aff[S] C YS .

## Affine Independence

A direction space of the affine subspace spanned by a system of k +1 points Po , . . . , Pk consists of 0 - weight linear combinations of these points . This linear space is generated by the differences Pi — P0 , . . . , Pk — P0 , therefore its dimension is at most k .

Definition . The points P0 , . . . , Pk are called affinely independent if any of the following equivalent conditions is fulfilled .

P0 , . . . , Pk span a k - dimensional affine subspace .

The vectors Pi — P0 , . . . , Pk — P0 are linearly independent .

A 0 - weight linear combination of P0 , . . . , Pk equals 0 only if all the coefficients are equal to 0 .

## Affine Coordinate Systems

An affine coordinate system on an affine space A = ( X , V , $ ) is given by the following data:

a point O G X , which will be the origin of the coordinate system;

and a basis ei , . . . , en of V .

Given an affine coordinate system , the coordinates of a point P are the real numbers ( x1 , . . . , xn ) for which

$ ( O , P ) = x1e1 + • • • + xnen .

Assigning to each point of X its coordinate vector gives an affine isomorphism between the affine space A and the affine space ARn .

Tangent Vectors at a Point , and the Tangent Bundle

A free vector v G V of an affine space ( X , V , $ ) has no given initial point or endpoint . It can be represented by any ordered pair of points ( P , Q ) for which $ ( P , Q ) = v . Sometimes we have to consider vectors with a given base point instead of free vectors . For example , forces in Newtonian mechanics are vector like quantities . However , to describe a diagram of forces properly , it is not enough to know the directions and the magnitudes of the acting forces , we have to know also the ( base ) points at which the forces act on a given body . For instance , if we push a wardrobe by a horizontal force , sufficiently large to overcome friction , then the wardrobe will slide horizontally if we push it close to the floor , but it may fall over if it is pushed at the top . The reason of this fact is that the torque of a force depends on the point at which the force acts on a body .

Definition . Let P G X be a point of an affine space ( X , V , T ) . A vector based at P or a tangent vector at P is a pair ( P , v ) , where v G V . The tangent space TpA ( or TpX ) of the affine space A at P is the set of all tangent vectors at P , i . e . , TPA = { ( P , v ) | v G V} .

Tangent vectors at P form a linear space with the operations

A ( P , v ) + p ( P , w ) = ( P , Av + pw ) .

"Forgetting the base point" is a linear isomorphism iP : TPA ^ V , iP ( P , v ) = v between TP A and V . If P and Q are two different points in X , then the linear isomorphism i - 1 o iP : TPA ^ TqA is called parallel transport between tangent spaces at P and Q .

Definition The disjoint union of all tangent spaces of an affine space is called its tangent bundle TA = |JPeX TPA = X x V . The map n: TA ^ X , which assigns to each tangent vector ( P , v ) its base point P is called the projection of the tangent bundle .

# 8.3 EUCLIDEAN SPACES

Definition . An affine space ( X , V , T ) is a Euclidean space if the linear space V is endowed with positive definite symmetric bilinear function ( , } , making it a Euclidean linear space .

Since every linear subspace W of V inherits a Euclidean structure by restricting the inner product ( , } onto W x W , affine subspaces of a Euclidean space are Euclidean spaces as well .

The Euclidean structure enables us to introduce metric notions like distance , angle , area , volume etc .

## Distance and the Isometry Group

Definition . The distance between two points P and Q of a Euclidean space ( X , V , T ) is d ( P , Q ) = ||T ( P , Q ) || .

Definition 1 . 3 . 20 . A metric space is a pair ( X , d ) , where X is a set , d: X x X ^ R is a functions , called distance function , satisfying the following axioms

d ( P , Q ) > 0 for all P , Q G X and d ( P , Q ) = 0 if and only if P = Q .

d ( P , Q ) = d ( Q , P ) for all P , Q G X ( symmetry ) .

d ( P , Q ) + d ( Q , R ) > d ( P , R ) for all P , Q , R G X ( triangle inequality ) . IS

Proposition . Every Euclidean space is a metric space with its distance function .

Proof . The triangle inequality follows from Corollary

d ( P , Q ) + d ( Q , R ) = ||T ( P , Q ) || + ||T ( Q , R ) \\ > ||T ( P , Q ) + T ( Q , R ) \\ = ||$ ( P , R ) || = d ( P , R ) .

Definition . A map y: X ^ Y between the metric spaces ( X , dx ) and ( Y , dy ) is said to be a distance preserving map or isometry if dy ( y ( P ) , y ( Q ) ) = dx ( P , Q ) for all P , Q G X . X

An isometry is always injective but not necessarily surjective .

Exercise . Give an example of an isometry y: X ^ X which maps a metric space ( X , d ) onto one of its proper subsets .

Definition . Bijective isometries of a metric space form a group with respect to composition . The group of bijective isometries of the metric space X is called the isometry group of X , and is denoted by Iso ( X ) .

Theorem . Every isometry of a Euclidean space En = ( X , V , T ) into itself is bijective . Isometries of En are affine transformations . An affine transformation T: X ^ X is an isometry if and only if the corresponding linear map L: V ^ V , for which L ( AlB ) = T ( A ) T ( B ) , preserves the norm of vectors .

A linear transformation of the Euclidean linear space V preserves the norm of vectors if and only if it preserves the dot product of vectors . If M is the matrix of L with respect to an orthonormal basis , then this property of L is also equivalent to the matrix equation

MMt = Mt M = I .

Definition . Linear transformations of a Euclidean space preserving the dot product are called orthogonal linear transformations . Matrices satis - fying equation are called orthogonal matrices .

Definition . An orientation preserving isometry or motion of a Eu - clidean space is an isometry , the associated linear transformation of which has positive determinant .

Orientation preserving isometries form a normal subgroup Iso+ ( En ) in the isometry group Iso ( En ) of the Euclidean space .

## The Angle Between Affine Subspaces

Angle between affine subspaces of the same dimension will be defined as the angle between their direction spaces . Direction spaces are linear subspaces , so we first define the angle between linear subspaces .

Definition . The angle Z ( e , f ) between two 1 - dimensional linear subspaces e = lin[v] and f = lin[w] of a Euclidean linear space V , is the smaller of a and n — a , where a is the angle between v and w .

Clearly , Z ( e , f ) is the unique angle in [0 , n / 2] for which

v , w

»s<Z<e . f ) ) = •

Proposition . Z is a metric on the projective space P ( V ) .

Proof . We prove only the triangle inequality , the rest is trivial . Let e = lin[v] , f = lin[w] , g = lin[z] be three 1 - dimensional subspaces and let us show that

Z ( e , f ) + Z ( f , g ) > Z ( e , g ) .

Assume that the vectors v , w and z have unit length . Denote by a , 0 and Y the angles between ( v , w ) , ( w , z ) and ( z , v ) respectively . Changing the direction of the vectors w and z if necessary , we may assume that a , 0 G [0 , n / 2] . By Corollary 1 . 2 . 58 , the Gram matrix of the vectors v , w , z has nonnegative determinant , thus

(

1 cos ( a ) cos ( y ) n cos ( a ) 1 cos ( 0 ) cos ( y ) cos ( 0 ) 1

= 1 + 2 cos ( a ) cos ( 0 ) cos ( y ) — cos2 ( a ) — cos2 ( 0 ) — cos2 ( y ) > 0 .

This is a quadratic inequality for cos ( y ) , which is fulfilled if and only if cos ( y ) is in the closed interval the endpoints of which are the roots of the polynomial

P ( t ) = ( 1 — cos2 ( a ) — cos2 ( 0 ) ) + 2cos ( a ) cos ( 0 ) t — t2 .

The roots of P are

cos ( a ) cos ( 0 ) ± \Jcos2 ( a ) cos2 ( 0 ) + 1 — cos2 ( a ) — cos2 ( 0 ) =

= cos ( a ) cos ( 0 ) ± ^ / ( 1 — cos2 ( a ) ) ( 1 — cos2 ( 0 ) ) = cos ( a ± 0 ) .

Since the cosine function is strictly decreasing on the interval [0 , n] , this implies

|0 — a| < y 0 a + 0 ,

in particular ,

Z ( e , f ) + Z ( f , g ) = a + 0 > y > min{Y , n — y} = Z ( e , g ) .

We define the angle between k - dimensional linear subspaces using the Pliicker embedding Grk ( V ) ^ P ( AkV )

Definition . The angle between two k - dimensional linear subspaces W2 , W2 of a linear space V is the angle between the 1 - dimensional linear subspaces lin[w} A • • • A wk] and lin[w2 A • • • A w2] , where w} , . . . , wk is a basis of W2 , w} , . . . , w| is a basis of W2 .

Definition . The angle between two affine subspaces of a Euclidean space is the angle between their direction spaces .

Since two different affine subspaces can have the same direction space , the angle function is not a metric on the affine Grassmann manifold AGrk ( A ) .

## 8.4 CARTESIAN COORDINATE SYSTEMS

Definition . An affine coordinate system given by the origin O and the basis vectors ( e 2 , . . . , en ) is a Cartesian coordinate system , if the basis ( e 2 , . . . , en ) is orthonormal . Jg

A Cartesian coordinate system defines an isometric isomorphism between any n - dimensional Euclidean space and the standard model ARn of the n - dimensional Euclidean space , where the inner product on Rn is the standard dot product .

Equations and Parameterizations of Affine Subspaces Linear subspaces

Let us first deal with equations and parameterizations of linear subspaces of a linear space V . A k - dimensional linear subspace W of V can be given by k linearly independent vectors a 2 , . . . , ak spanning W . Then a vector x G V belongs to W if and only if x can be written as a linear combination of the vectors a2 , . . . , ak .

This way , the mapping

r: Rk ^ V , r ( x1 , . . . , xk ) = x1a1 + ■ ■ ■ + xkak

maps Rk bijectively onto W . The map r is called a ( linear ) parameterization of W . A linear subspace has many linear parameterizations , since it has many different bases .

We can also define a linear subspace as the kernel of a linear map , i . e . , as the set of solutions of a system of linear equations .

For this purpose , observe that x is in W if and only if ai , . . . , ak , x are linearly dependent . By Proposition linear dependence is equivalent to the equation

x A a1 A ■ ■ ■ A ak = 0 .

This is called the equation of the linear subspace W . This equation requires the vanishing of a ( k + 1 ) - vector . If dim V = n , then the space of Ak+1 V has dimension ( k+J . If we fix a basis in V , then the equation of W becomes equivalent to a system of ( k+ J equations requiring the vanishing of all the coordinates of the ( k + 1 ) - vector on the left - hand side . Since the coordinates are linear functions of x , all the equations in the system are linear . Obviously , these equations are not linearly independent in general . By the dimension formula any maximal linearly independent subsystem of this system of equations contains exactly ( n — k ) equations . Thus there are many different ways to convert the equation of a linear subspace to an independent system of ( n — k ) linear equations .

In the special case k = n — 1 of linear hyperplanes , however , we have only one linear equation on the coordinates , and it is uniquely determined by W up to a scalar multiplier . If we introduce an orientation and a Euclidean structure on V , then the equation of W can be written also as

( x , * ( ai A ■ ■ ■ A a,,_i ) } = 0 .

This way , W contains all vectors orthogonal to * ( a1 A ■ ■ ■ A an - 1 ) . This last vector is a normal vector of W .

Affine subspaces

Assume that we want to parameterize or write the equation of a k - dimensional affine subspace W spanned by k +1 affinely independent points a0 , . . . , ak . It is not difficult to parameterize W , that is to obtain it as the image of a bijection r: Rk ^ W . We know that a point belongs to W if and only if it is an affine combination of the points a0 , . . . , ak , consequently , the mapping

r ( x1 , . . . , xk ) = ( 1 — ( x1 + ■ ■ ■ + xk ) ) a0 + x1a1 + ■ ■ ■ + xkak

is a parameterization of W .

To write the equation of an affine subspace we use a trick rooted in projective geometry . Vectorize our affine space by choosing an origin . This identifies the space with AV , where V is a linear space . Consider the linear space ( V} = R ==> V and embed V into ( V} by the mapping v ==> v , where v = ( 1 , v ) . The image V of V is the translate of the linear subspace {0} ==> V by the vector ( 1 , 0 ) , thus , it is an affine subspace of ( V} , just as the image W of W . Denote by ( W} the linear subspace of ( V} spanned by W

( W} = lin[WW] = aff [WW U { ( 0 , 0 ) }] .

( W} is a ( k + 1 ) - dimensional linear subspace and uniquely determines W as the projection of W = ( W} n V onto V . This way , the map W → ( W} gives an embedding of the affine Grassmann manifold AGr^ ( V ) into the linear Grassmann manifold Grk+i ( ( V} ) .

Using this picture , x belongs to the affine subspace W if and only if X G ( W} . Since the linear subspace ( W} is spanned by the vectors ao , . . . , ak , the equation of the affine subspace W is

x a a0 a ■ ■ ■ a ak = 0 .

Introducing a basis in V , this equation can be converted into a system of equations , in which each equation requires the vanishing of a linear combination of the coordinates of X . This means that every affine subspace can be defined by a system of inhomogeneous linear equations of the form

$co + c1x1 + \quad + cnxn = 0$ .

The number of independent equations that define W is ( n+1 ) - ( k+1 ) = n - k . In the case of a hyperplane

In a Cartesian coordinate system , the coefficients describe the hyperplane as follows . N = ( ci , . . . , cn ) is a normal vector of the hyperplane , and the hyperplane goes through the point — coN / ||N||2 . The constant c0 can be computed from any point p0 lying in the hyperplane as c0 = — <N , po> .

# Equations of Spheres

Definition Let E be a ( k+1 ) - dimensional subspace of an n - dimensional Euclidean space , O G E be a point , r > 0 be a positive number . Then the k - dimensional sphere or shortly the k - sphere in E centered at O with radius r is the set of those points in E the distance of which from O is r . A hypersphere is an

( n — 1 ) - dimensional sphere . The case r = 0 is considered to be a degenerate case , when the sphere degenerates to a point .

In the non - degenerate case , a k - sphere determines its ( k + 1 ) - plane , its center and its radius uniquely .

To write the equations of k - spheres introduce a Cartesian coordinate system on the space . This identifies our space with Rn .

Consider first hyperspheres ( E = Rn ) . If o is the center of the sphere , then the equation of the sphere is

||x — o||2 — r2 = ||x||2 + <—2o , x> + ( ||o||2 — r2 ) = 0 .

This proves that every hypersphere can be defined by an equation of the form

a||x||2 + <b , x> + c = 0 ,

where a , c G R , b G Rn .

Definition . Equations of the form will be called hypersphere equations or shortly sphere equations .

The solution set of a sphere equation is described in the following proposition .

Proposition . Let S = {x | a||x||2 + <b , x> + c = 0} be the set of solutions of the sphere equation . Then we have the following cases .

If the equation is trivial , i . e . , a = c = 0 , b = 0 , then the solution set is the whole space Rn .

If the equation is contradictory , i . e . , a = 0 , b = 0 , but c = 0 , then S = 0 .

If the equation is inhomogeneous linear , that is a = 0 , but not trivial and not contradictory , that is b = 0 , then S is a hyperplane .

• If the equation is quadratic , i . e . , a = 0 , then S depends on the sign of the discriminant d = ||b||2 — 4ac .

If d> 0 , then S is a hypersphere centered at —b / ( 2a ) with radius yfd / \2a\ .

If d = 0 , then S is a degenerated hypersphere of radius 0 which contains a single point —b / ( 2a ) .

If d < 0 , then S = 0 .

Proof . The linear case is simple , the quadratic case follows from the equivalent rearrangemen of equation

Proposition . The intersection S of a hypersphere of radius r centered at O and a hyperplane <= in an n - dimensional Euclidean space is described as follows . Let the orthogonal projection of O onto <= be O' and denote by d the distance OO' . Then if r > d , then S is an ( n — 2 ) - sphere in <= with radius \ / r2 — d2 , and center at O' . S consists of the single point O' when r = d and S = 0 when r > d .

The proof is a corollary of the Pythagorean theorem , since for any P in ≤ , the triangle POO'A is a right triangle .

Corollary . The intersection of a finite system of hyperspheres and hyperplanes is either empty or an affine subspace or a k - sphere . Proof . From an algebraic viewpoint , the intersection is the set of solutions of a finite number of sphere equations . If there is a quadratic sphere equation in the system , then subtracting a suitable multiple of it from the other equations , we can make all the other equations linear , so we may assume that there is at most one hypersphere in the family .

If there is no hypersphere at all , the statement follows from Corollary . If there is a hypersphere in the family , then we can prove the statement inductively , using the previous proposition .

To write the equation of a k - sphere , we use an embedding of Rn into a larger linear , space analogous to the affine embedding V ==> ( V ) = R ==> V that was used when we wrote equations of affine subspaces .

To simplify notation , denote the Euclidean linear space Rn by V . Set ( ( V ) ) = R ==> V ==> R . Embed V into ( ( V ) ) with the map

v i ==> - v , where v = ( 1 , v , ||v||2 ) .

Because of the quadratic term in the last coordinate , this is not an affine embedding . Its image V is a paraboloid . The advantage of this embedding is that a linear equation on the coordinates of v is a hypersphere equation on v .

Proposition . Let a0 , . . . , ak be ( k + 1 ) > 2 affinely independent points . Then there is a unique ( k — 1 ) - sphere through these points and it can be defined by the equation

x a a0 A • • • A ak = 0 .

Proof . The ( k — 1 ) - sphere must be in the unique k - plane E spanned by the points . If the points were contained in two different ( k — 1 ) - spheres of E , then they would be contained in their intersection , which is a ( k — 2 ) - sphere . This would contradict affine independence of the points , since every ( k — 2 ) - sphere is contained in a ( k — 1 ) - plane . Thus , uniqueness is proved .

To complete the proof , we should check that the set given by the equation defines a ( k — 1 ) - sphere passing through the given points . Let S be the set of solutions

Since equivalent to a system of linear equations on the coordinates of X , which is a system of sphere equations on x . Thus S is either empty , or an affine subspace , or a sphere .

The set S cannot be empty , since it contains the points a0 , . . . , ak as

a a a0 a • • • a ak = 0 .

This also gives a lower bound on the dimension of S . If it is an m - sphere , then m > k — 1 , if it is an m - plane , then m > k .

If x € S , then x is a linear combination of the vectors ao , . . . , ak , hence

( 1 , x , ||x||2 ) = ao ( 1 , ao , ||ao^2 ) 4 4 ak ( 1 , afc , ||afc||2 )

for some a0 , . . . , ak € R . This equation splits into three components

1 = ao + • • • + ak , x = aoa0 + • • • + ak ak ,

||x||2 = ao||ao||2 +

The first two equations show that x must be an affine combination of the points ao , . . . , ak . Thus , S C E , and we have only two possibilities left . Either S is a ( k — 1 ) - sphere , in which case we are done , or S = E . The latter case can be excluded by showing that the midpoint ( ao + ai ) / 2 is not in S . If it were in S , then by affine independence of the aj's , the only possible choice for the aj's would be ao = a1 = 1 / 2 and aj = 0 for 2 < i < k , however , the third equation is not fulfilled with these coefficients as > 0 .

This proves the proposition .

Denote by SPHk ( V ) the set of all k - spheres in the Euclidean linear space V . As a corollary of the previous proposition , we can construct an embed - ding SPHk ( V ) ^ Gr1 ( Ak+1 ( ( V ) ) ) in the following way . Given a k - sphere S , choose k affinely independent points ao , . . . , ak from it , and assign to S the 1 - dimensional linear space spanned by the ( k + 1 ) - vector ao A • • • A ak € Ak+1 ( ( V ) ) .

Exercise . Prove that the center of the sphere defined by the point aoao + • • • + akak , where the coefficients are obtained as the solution of the following system of linear equations

ao + • • • + ak = 1

ao ( a — ao , ao ) 4 4 ak ( a — ao , ak ) = 11 a j |2 — ||ao||2 for i = 1 , . . . , k . Q

Ordinary Differential Equations

**Definition .** Let U C $R^n$ be an open subset of $R^n$ . A vector field over U is a map **F**: U ^ $TR^n$ such that **F ( p )** G $T_pR^n$ for all **p** G U . IS

A tangent vector based at **p** is a pair **( p , v ) ,** where **v** G $R^n$ , therefore , **F ( p )** must have the form **( p** , F **( p ) ) .** As the function F: U ^ $R^n$ obtained from **F** by ignoring the base points determines **F** uniquely , we can define a vector field uniquely by a mapping from U to $R^n$ .

**Definition .** Let U C $R^n$ be an open subset of $R^n$ , F: U ^ $R^n$ be a vector field on U . A ( first order autonomous vector valued ) ordinary differ - ential equation with right - hand side F is the problem of finding differentiable parameterized curves , i . e . , differentiable maps 7: I ^ U satisfying the equation

Y' ( t ) = F ( Y ( t ) ) for all t G I ,

where I is a finite or infinite interval in R . Solutions of the problem are called the integral curves of the differential equation or that of the vector field F . F is also called the right - hand side of the differential equation .

More generally a kth order vector valued ordinary differential equation is given by a map F: U x $( R^n )^{k-1}$ ^ $R^n$ and is posing the problem to find k - times differentiable curves Y : I ^ U satisfying

$Y^{(k)}$( t ) = F ( Y ( t ) , Y' ( t ) , . . . , $Y^{(k-1)}$( t ) ) for all t G I .

Despite its more general form , every kth order differential equation is equiva - lent to a first order one . The equivalent problem is to find a curve ( Y , $n_1$ , . . . , $n_{k-1}$ ) : I ^ U x $( R^n )^{k-1}$ which satisfies the first order differential equation

$(Y, \eta_1, \ldots, \eta_{k-1})'(t) = (\eta_1(t), \ldots, \eta_{k-1}(t), F(Y(t), \eta_1(t), \ldots, \eta_{k-1}(t)))$.

The adjective autonomous refers to the circumstance that the right - hand side F depends only on the position Y ( t ) but not on other parameters , say t . For example , a time dependent non - autonomous differential equation has the form

$$Y'(t) = F(t, Y(t))$$

where F: R x U ^ $R^n$ is the time dependent right - hand side . However , such a differential equation can also be rephrased as an autonomous differential equation . Indeed , it is equivalent to finding curves ( T , Y ) : I ^ RxU satisfying

$(T, Y)'(t) = (1, F(T(t), Y(t)))$.

For these reasons , we shall summarize here the fundamental theorems of ordinary differential equations only for autonomous first order systems .

**Definition .** A maximal integral curve of a differential equation is a solution which cannot be extended to a larger interval as a solution of the differential equation .

**Theorem** ( Existence and Uniqueness of Solutions ) . If the right - hand side F: U ^ $R^n$ of a first order differential equation is continuously differentiate , then for any **p** G U , there is a unique maximal integral curve $Y_P$ : ( $a_p$ , $b_p$ ) ^ U such that —TO < $a_p$ < 0 < $b_p$ < +TO and $Y_p$ ( 0 ) = **p .**

Remark that for continuous right - hand side only the existence part of the theorem is true , uniqueness may fail .

**Theorem** ( Smooth Dependence on the Initial Condition ) . Suppose that the right - hand side of a first order ordinary differential equation is smooth . Then the set W = **{ ( p ,** t ) | **p** G U , t G ( $a_P$ , $b_p$ ) is an open subset of U x R , and the map $: W ^ U , **$ ( p ,** t ) = $Y_P$ ( t ) is a smooth map .

In other words , the point at which we arrive at after traveling for time t along an integral curve starting at **p** depends smoothly on **p** and t . As a corollary , for any t G R , the set $W_t$ = **{p | ( p ,** t ) G W} is an open subset of U , and the map $$_t$: $W_t$ ^ U , **$$_t$ ( p ) = $ ( p ,** t ) is smooth . If ( **p** , to ) G W and **q** = $\gamma_p$ ( to ) , then the map ( $a_p$ — $t_0$ , $b_p$ — $t_0$ ) ^ U , t ^ $\gamma_p$ ( t + to ) is a maximal integral curve starting at **q ,** therefore , $\gamma_{\mathbf{q}}$ ( t ) = $\gamma_p$ ( t + to ) , **q** G $W_{-to}$ and $\$_{-to}$ **( q )** = **p .** In conclusion , $\$_t$: $W_t$ ^ $W_{-t}$ is a diffeomorphism for all t G R .

**Definition .** The family $\{\$_t\}_t$gR is called the flow or one - parameter family of diffeomorphisms of the ordinary differential equation or the vector field F .

The flow satisfies the following group property . If **( p** , t ) G W and **( $\$_t$ ( p** ), s ) G W , then **( p ,** t + s G W ) and

$\$$t+s **( p )** = $\$_s$ ( $\$$t **( p ) ) .**

The typical reason why an integral curve cannot be extended to [0 , ±TO ) is that it is running out to infinity or to the boundary of U within a finite time .

**Theorem** ( Unboundedness of Maximal Solutions in Time or Space ) . If for a maximal integral curve $\gamma_p$ : ( $a_p$ , $b_p$ ) ^ U , $a_p$ = —TO ( or $b_p$ = +TO ) , then the trace Y ( ( $a_p$ , 0] ) ( or Y ( [0 , $b_p$ ) ) , respectively , ) cannot be covered by a bounded closed subset of U .

## 8.5 LINEAR DIFFERENTIAL EQUATIONS

Denote by $R^{nxm}$ the linear space of n x m matrices and identify $R^n$ with the space $R^{nx1}$ of column vectors . A linear differential equation is an equation of the form

$$\mathbf{x}' = A \cdot \mathbf{x} ,$$

where **x**: I ^ $R^n$ is an unknown column vector valued function defined on the interval I , A: I ^ $R^{nxn}$ is a given matrix valued function , • denotes matrix multiplication . As the matrix A typically depends on t , linear differential equations are usually non - autonomous .

Of course , the fundamental theorems on ordinary differential equations are true also for linear differential equations , but linearity has some additional consequences not true in the general case . These are sum up in the following theorem .

Theorem . If A: I ^ $R^{nxn}$ is smooth , to G I is a given initial point and **x**o G $R^n$ is an arbitrary initial value , then there is a unique solution **x**: I ^ $R^n$ for which **x** ( $t_0$ ) = $\mathbf{x_0}$ **.**

Recall that for a general differential equation , solution exist only in a certain , maybe small neighborhood of the initial point $t_0$ . In the linear case , however , solution exists on the whole interval I . Solutions of the linear differential equation form an n - dimensional linear space with respect to pointwise addition and multiplication by real numbers . This property is a characterization of linearity of a differen - tial equation . In the special case when A is a constant matrix , solution of with initial condition **x** ( to ) = **x**o can be written explicitly as

$$\mathbf{x}(t) = e^{\mathbf{A}(t - to)}\mathbf{x_0},$$

where the exponential of a matrix M is defined as $e^M$ = ^ᶜjfi₌**0**
pp - .

Systems of Total Differential Equations

Systems of total differential equations are multivariable generalizations of the non - autonomous differential equation

**Definition** A system of total differential equations is an equation of the form G' **( u )** = F **( u** , G **( u ) )**

for an unknown multivariable function G , where F: 0 ^ $R^{nx}$m is a given matrix valued function on an open subset of 0 C $R^m$ x $R^n$ . G: U ^ $R^n$ is a solution of the system , if it is defined on an open subset U of $R^m$ , its graph **{ ( u ,** G **( u )** | **u** G U} is contained in 0 , and ( 1 . 22 ) is satisfied by all **u** G U . A system of total differential equations is integrable if for any **(** $\mathbf{u_0}$ **,** $\mathbf{v_0}$ **)** G 0 , there is a solution G of the system satisfying the initial condition G **(** $\mathbf{u_0}$ **)** = $\mathbf{v_0}$ **.** is If G is a solution with given initial value G **(** $\mathbf{u_0}$ **)** = **v**o , **( e**i , . . . , $\mathbf{e_m}$ **)** is the standard basis of Rm , then for any 1 < i < m , the curve qi ( t ) = G ( u0 + tej ) satisfies the ordinary differential equation

Yi ( t ) = F **( u**o + t**e**i , Yi ( t ) ) • **e**i with the initial condition $Y_i(0) = \mathbf{u_0}$ . ( In this equation , vectors are column vectors , "•" denotes matrix multiplication . ) Solving these differential equa - tions we can compute the values of G in a neighborhood of $\mathbf{u_0}$ along segments through $\mathbf{u_0}$ parallel to one of the coordinate axes . As any point of $R^m$ can be connected to $\mathbf{u_0}$ by a broken line the segments of which are parallel to one of the coordinate axes , iterating this process we can compute the val - ues of G in a small neighborhood of $\mathbf{u_0}$ . This proves that the solution of a system of total differential equations with a given initial condition is unique in a neighborhood of the initial point . However , the solution may not exist . When m > 2 , there are many ways to choose the broken line connecting $\mathbf{u_0}$ to a point $\mathbf{u_1}$ nearby , and it can happen that computing $G(\mathbf{u_1})$ with the help of different broken lines we get different values . In such a way the system has no solution with the given initial value and the system is not integrable . Frobenius' Theorem gives a necessary and sufficient condition for the in - tegrability of a system of total differential equations . The easiest way to paraphrase the condition is that the system is integrable if and only if it does not contradict to Young's Theorem Compute what this means in terms of formulae . Let $F_j^i(\mathbf{u}, \mathbf{v})$ be the matrix element of $F(\mathbf{u}, \mathbf{v})$ in the ith row and jth column . If $G(\mathbf{u}) = (G^1(\mathbf{u}), \ldots, G^n(\mathbf{u}))^T$ is a solution of the system , then

$$d_j G^i(\mathbf{u}) = F_j^i(\mathbf{u}, G(\mathbf{u})) \text{ for all } 1 < i < n,\ 1 < j < m .$$

Differentiating this equality with respect to the kth variable we obtain n

$$d_k d_j G^i(\mathbf{u}) = d_k F_j^i(\mathbf{u}, G(\mathbf{u})) + \Upsilon\, d_{m+s} F_j^i(\mathbf{u}, G(\mathbf{u}))\, d_{fc} G^s(\mathbf{u})$$

s = 1 n

$$= d_k F_j^i(u, G(u)) + \Upsilon\, d_{m+s} F_j(u, G(u))\, F_k(u, G(u)) .$$

s = 1

By Young's Theorem , the right hand side of this equality should not change if we flip the role of j and k .

**Theorem** ( Frobenius' Theorem ) . The system of total differential equations is integrable if and only if

$$d_k F_j + \sum_{s=1}^{n} d_{m+s} F_j W_k = d_j F_k + \sum_{s=1}^{n} d_{m+s} F_k F /$$

holds on Q for all 1 < i < n , and 1 < j , k < m .

A geometrical version of Frobenius' Theorem

**Check your Progress 1**

Discuss Geometry

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Discuss Linear Differential Equtions

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## 8.6 LET US SUM UP

In this unit we have discussed the definition and example of Geometry , Euclidean Spaces , Cartesian Coordinate Systems , Linear Diffrential Equations

## 8.7 KEYWORDS

Geometry … Affine Geometry… Affine Spaces, In traditional axiomatic treatment of Euclidean geometry

Euclidean Spaces… An affine space ( X , V , T ) is a Euclidean space if the linear space V is endowed with positive definite symmetric bilinear function ( , } , making it a Euclidean linear space

Cartesian Coordinate Systems… An affine coordinate system given by the origin O and the basis vectors ( e 2 , . . . , en ) is a Cartesian coordinate system

Linear Diffrential Equations….. Denote by $R^{nxm}$ the linear space of n x m matrices and identify $R^n$ with the space $R^{nx1}$ of column vectors . A linear differential equation is an equation of the form

## 8.8 QUESTIONS FOR REVIEW

Explain Geometry , Linear Diffrential Equations

## 8.9 SUGGESTED READING

Differential Geometry, Differential Geometry & Application, Introduction to Defferential Geometry, Basic of Differential Geometry.

## 8.10 ANSWERS TO CHECK YOUR PROGRESS

Geometry

Linear Diffrential Equations

( Answer for Check your Progress - 1 Q )

# UNIT-9 : LINEAR ALGEBRA

**STRUCTURE**



## 9.0 OBJECTIVES

After studying this unit , you should be able to:

- Understand about Linear Algebra
- Orientation of Linear Space ,
- Exterior Powers ,
- Euclidean linear spaces

## 9.1 INTRODUCTION

Differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces Mathematical analysis of curves and surfaces had been developed to answer some of unanswered questions that appeared in calculus like the reasons for relationships between complex shapes and curves , series and analytic functions , Linear Algebra , Orientation of Linear Space , Exterior Powers , Euclidean linear spaces

## 9.2 LINEAR ALGEBRA

### Linear Spaces and Linear Maps

Definition A set V is a linear space , or vector space over R if V is equipped with a binary operation + , and for each A G R , the multiplication of elements of V with A is defined in such a way that the following identities are satisfied:

( x + y ) + z = x + ( y + z ) ( associativity ) ;

3 0 G V such that x + 0 = x for all x G V;

V x G V 3 — x G V such that x + ( — x ) = 0;

x + y = y + x ( commutativity ) ;

A ( x + y ) = Ax + Ay;

( A + p ) x = Ax + px;

( Ap ) x = A ( px ) ;

1x = x . m

Definition . A linear combination of some vectors xi , . . . , xk G V is a vector of the form

A1x1 + ••• + Ak xk ,

where the Aj's are real numbers . The vectors x1 , . . . , xk G V are linearly in - dependent if the linear combination can be 0 only if all of the coefficients Aj vanish . A basis of V is a maximal set of linearly independent vectors . IS

It is known that any two bases of a linear space have the same cardinality .

Definition . The dimension dim V of the linear space V is the cardinality of a basis .

Definition A map L: V ^ W between the linear spaces V and W is said to be linear if

L ( Ax + py ) = A L ( x ) + pL ( y )

for any x , y G V and A , p G R . A linear isomorphism is a bijective linear map . Two linear spaces are isomorphic if there is a linear isomorphism between them .

Linear spaces as objects and linear transformations as morphisms form a category . Two linear spaces are isomorphic if and only if they have the same dimension . The automorphism group of a linear space V is called the general linear group of V and it is denoted by GL ( V ) .

If V is an n - dimensional linear space , and ( e1 , . . . , en ) is a basis of V , then any vector x G V can be written uniquely as a linear combination x = x1ei + • • • + xnen of the basis vectors . The numbers ( x1 , . . . , xn ) are called the coordinates of x with respect to the basis ( ei , . . . , en ) .

The indices of the coordinates are not exponents , they are just upper indices . The reason why it is practical to use both upper and lower indices is the ob - servation that if we position the indices properly in a linear algebraic formula , then usually summations go exactly over those indices that appear twice in a term , once as a lower index , once as an upper one . Therefore , if we take care of the right positioning of the indices , summation signs show redundant information and can be supressed . This leads to Einstein's convention which suggests us to position the indices properly and omit the summation signs . It is a rule for correct index positioning that if a single index appears on one side of an equation , then the same index must appear as a single index at the same ( upper or lower ) position on the other side as well . In this book , we shall pay attention to index positioning , but we shall not omit the summation signs .

**Notes**

Working with coordinates , linear transformations are represented by matri - ces . Let L: V ^ W be a linear map from the n - dimensional linear space

to the m - dimensional linear space W . Choose a basis ( e1 , . . . , en ) for

and a basis ( f1 , . . . , fm ) for W . Write L ( ej ) as a linear combination L ( ej ) = 11f1 + • • • + / mfm . Arranging the coefficients / j into an m x n matrix

we obtain the matrix of L with respect to the bases ( e1 , . . . , en ) and ( f^ . . . , fm ) If we arrange the coordinates of x with respect to the basis ( e1 , . . . , en ) into a column vector [x] , then the column vector [L ( x ) ] of the coordinates of L ( x ) with respect to the basis ( f1 , . . . , fm ) can be computed by the matrix mul - tiplication [L ( x ) ] = [L][x] . For an endomorphism of V , we usually use the same basis for V and W = V .

Examples .

Recall that Rn denotes the set of n - tuples of real numbers Rn = { ( X1 , . . . , Xn ) |Xj € R} .

If x = ( x1 , . . . , xn ) and y = ( y1 , . . . , yn ) are two elements of Rn and A € R is a real number , then we define the sum and difference of x and y and the scalar multiple of x by

x ± y = ( x1 ± y1 , . . . , xn ± yn ) , Ax = ( Ax1 , . . . , Axn ) .

It is clear that Rn is an n - dimensional linear space over the field of real numbers with respect to the operations defined above .

Let ei denote the vector ( 0 , . . . , 0 , 1 , 0 . . . 0 ) , the only non - zero coordinate of which is the ith one , being equal to 1 . The n - tuple ( ei , . . . , en ) is a basis of Rn called the standard basis of Rn .

Let V and W be linear spaces , Hom ( V , W ) be the set of all linear maps from V to W . Hom ( V , W ) becomes a linear space if for A , B G Hom ( V , W ) and A G R , we define the maps A + B and A A by

( A + B ) ( x ) = A ( x ) + B ( x ) , ( AA ) ( x ) = A ( A ( x ) ) .

A linear subspace of a linear space V is a nonempty subspace W C V , which contains all linear combinations of its elements . Linear subspaces of a linear space are linear spaces themselves . The dimension of a linear subspace W of V is less than or equal to dim V . If V is finite dimensional , then dim W = dim V holds only if V = W .

The intersection of an arbitrary family of linear subspaces is a linear subspace , therefore , for any subset S C V there is a unique smallest linear subspace among all linear subspaces containing S . We shall call this linear subspace the linear subspace spanned or generated by S , or simply the linear hull of S . We shall denote the linear hull of S by lin[S] .

Definition . The Grassmann manifold of k - dimensional linear sub - spaces of the linear space V is the set Grk ( V ) of all k - dimensional sub - spaces of V . In the special case k = 1 , P ( V ) = Gr1 ( V ) is also called the projective space associated to V . Later we shall introduce a topology and a manifold structure on Grk ( V ) . Then the name Grassmann manifold will be justified .

If L G Hom ( V , W ) is a linear transformation , then the image im L = {L ( v ) | v G V} of L is a linear subspace in W , and the kernel ker L = {v G V | L ( v ) = 0} of L is a linear subspace in V . The rank rk L of L is the dimension of im L .

For an element v G V of a linear space V , we define translation Tv by v as the map Tv: V ^ V , Tv ( x ) = x + v . Translations by a non - zero vector are not linear transformations . If W is a linear subspace of V then two translates of W are either equal or disjoint . The set V / W = {Tv ( W ) | v G V} of translates of W carries a linear space structure defined by ATV1 ( W ) + pTV2 ( W ) = T\Vl+MV2 ( W ) . ( Check that the definition is correct . ) The linear space V / W is called the factor space of V with respect to the subspace W .

The surjective linear map L: V ^V / W , v ^ Tv ( W ) is the factor map .

Every linear map L: V ^ W is a factor map L: V ^ V / ( ker L ) = imL onto its image . The dimension of the factor space or the image of a linear space can be computed by the formula dim ( V / W ) = dim ( V ) — dim ( W ) , rk L = dim ( im L ) = dim ( V ) — dim ( ker L ) .

The linear space V* = Hom ( V , R ) consisting of the linear functions on V is the dual space of V . Assigning the dual space V* to a linear space V is a contravariant functor of the category of linear spaces into itself . This functor assigns to a linear map L: V ^ W the adjoint map L*: W* ^ V* , defined by L* ( 1 ) = l o L , where l G V* .

If ( ei , . . . , en ) is a basis of V , then the linear functions e1 G V* , ( i = 1 , . . . , n ) , defined by the equalities el ( ej ) = <= j , where

is the Kronecker delta symbol , form a basis of V* . This basis is called the dual basis of the basis ( ei , . . . , en ) .

We remark that the Kronecker delta symbol 5j is also denoted by 5j and 5lj . In formulae involving the Kronecker delta symbol , we always position the indices as dictated by the Einstein convention .

Though for a finite dimensional linear space V , the dual space V* has the same dimension as V , there is no natural isomorphism between these two spaces . In other words , the identical functor on the category of finite dimen - sional linear spaces is not naturally isomorphic to the dual space functor . On the other hand , there is a natural transformation from the identical functor to the double dual space functor , given by the embeddings : V ^ V** , ( T ( v ) ) ( 1 ) = l ( v ) , where v G V , l G V* . The restriction of this natural trans - formation onto the category of finite dimensional linear spaces is a natural isomorphism .

With the help of this natural isomorphism , elements of a finite dimensional linear space V can be identified with elements of V** . With this identification , the dual basis of the dual basis of a basis of V will be equal to the original basis .

Determinant of Matrices and Linear Endomorphisms

Let us denote by Sn the group of all permutations of the set {1 , . . . , n} . For a permutation a G Sn , we denote by sgna the sign of the permutation a , n

a ( j ) — a ( i ) r T

— — € {—1 , 1} .

j— *

1<i<j<n

Exercise . Show that the sign of a permutation is always equal to ±1 .

Prove that sgn ( a1 o a2 ) = sgn ( a1 ) • sgn ( a2 ) .

Some important properties of the determinant are summarized in the following proposition .

Proposition If all but one columns of a square matrix are fixed , , then the determinant is a linear function of the varying column . This means that if we denote by ( ai , . . . , an ) the square matrix with column vectors a1 , . . . , an € Rn , then

det ( a1 , . . . , Aaj + paj , . . . , an ) — A det ( a1 , . . . , aj , . . . , an ) + p det ( a1 , . . . , aj , . . . , an ) .

When we permute the columns of a square matrix the determinant is multiplied by the sign of the permutation , i . e . ,

det ( aCT ( i ) , . . . , aCT ( „ ) ) = sgn a det ( a1 , . . . , an ) for all a € Sn .

The value of the determinant does not change if an arbitrary multiple of a column is added to another column .

The determinant of a matrix vanishes if and only if its columns are lin - early dependent .

A square matrix A and its transposition AT , that is the reflection of A in the main diagonal , have the same determinant

det A = det AT .

As a consequence , properties also if the columns are replaced by rows .

Notes

The determinant of an upper or lower triangular matrix is the product of the diagonal elements .

Recall that the product AB of two n x n matrices A and B is also an n x n matrix . The determinants of A , B and AB are related to one another as follows .

Proposition det ( AB ) = det ( A ) • det ( B ) .

Proposition . Let L: V ^ V be a linear endomorphism of the linear space V . Choose two bases ( ei , . . . , en ) and ( fi , . . . , fn ) of V and consider the matrices [L]e = ( lj ) i<j , j<n and [L]f = ( j ) i<j , j<n with respect to these bases respectively . Then det ( [L]e ) = det ( [L]f ) .

Proof . Let S: V ^ V be the invertible linear endomorphism which takes the basis vector ej to the basis vector fi for all i . Denote by [S]e = ( sj ) i<i , j<n the matrix of S with respect to the basis ( e1 , . . . , en ) . ThenL ( ej ) = E lj ej , L ( fj ) = E lj fj , S ( ej )

j=1and Comparing the coefficients we obtainICC'j sk = Y1Y . 'jj f°r all 1 < i , k <

j = 1 fc=1which means that [L]f [S]e = [S]e[L]e . Taking the determinant of both sides we obtain

det ( [L]f ) det ( [S]e ) = det ( [S]e ) det ( [L]e ) .

Since the columns of [S]e are linearly independent , as they are the coordinate vectors of the basis vectors fi with respect to the basis ( e1 , . . . , en ) , the deter - minant of [S]e is nonzero . Thus , equation implies the proposition . □

Definition . The determinant of a linear endomorphism L : V ^ V is the determinant of the matrix of L with respect to an arbitrary basis of V .

IS The definition is correct according to the previous proposition .

Definition . The complexification C <g ) V of a linear space V over R is the set of formal linear combinations v + iw , where v , w G V . IS

C ( g > V is a linear space of dimension 2 dim V , which contains V as a linear subspace . Elements of the complexification can also be multiplied by complex numbers as follows

( x + iy ) ( v + iw ) = ( xv — yw ) + i ( xw + yv ) .

A linear endomorphism of L: V ^ V can be extended to the complexification by the formula L ( v + iw ) = L ( v ) + iL ( w ) .

Definition . A non - zero vector z = v+iw = 0 of the complexification of a linear space V is an eigenvector of the linear endomorphism L: V ^ V if there is a complex number A G C such that L ( z ) = Az . The number A is called the eigenvalue corresponding to the eigenvector z . A complex number is an eigenvalue of L if there is an eigenvector to which it corresponds .

Proposition . A complex number A is an eigenvalue of the linear transformation L: V ^ V if and only if det ( L — A idy ) = 0 . An eigenvalue A is real if and only if there is an eigenvector in V with eigenvalue A .

Definition . The characteristic polynomial of a linear endomorphism L: V ^ V is the polynomial pL ( A ) = det ( L —A idV ) . Similarly , the character - istic polynomial of an n x n matrix A is the polynomial pa ( A ) = det ( A — A / „ ) , where In is the n x n unit matrix . The algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic polynomial .

The coefficients of the characteristic polynomial of a matrix A can be ex - pressed as polynomials of the matrix elements . They can also be expressed in terms of the eigenvalues Ai , . . . , An using the factorization

PA ( A ) = ( A1 — A ) • • • ( An — A ) .

Comparing these expressions we can relate elementary symmetric polynomi - als of the eigenvalues to some matrix invariants . For example , the constant term of the characteristic polynomial of the matrix A is

det A = pa ( 0 ) = Ai • • • An .

The coefficient of ( —A ) n - i in pa ( A ) is equal to the sum of the diagonal elements of the matrix and also to the sum of the eigenvalues .

Definition . The trace of a matrix A = ( a^ ) i<i , j<n is the number

n

tr A = ^2 au ,

i= 1

which is also equal to the sum of the eigenvalues of A . The trace of a linear endomorphism is the trace of its matrix with respect to an arbitrary basis .

## 9.3 ORIENTATION OF A LINEAR SPACE

Definition . Let ( vi , . . . , vk ) and ( wi , . . . , wk ) be two ordered bases of a linear space V . We say that they have the same orientation or they define the same orientation of V , if the k x k matrix ( aj ) defined by the system of equalities

k

vi = aj Wj for i = 1 , 2 , . . . , k

3=1

has positive determinant .

"Having the same orientation" is an equivalence relation on ordered bases , and there are two equivalence classes . A choice of one of the equivalence classes , the elements of which will be called then positively oriented bases , is an orientation of V .

Definition . The standard orientation of Rn is the orientation defined

i

by the ordered basis e1 , . . . , en , where ei = ( 0 , . . . , 0 , 1 , 0 . . . 0 ) .

IS

**Tensor Product**

Definition . Let V , W and Z be linear spaces . A map B: V xW ^ Z is said to be a bilinear map if it is linear in both variables , i . e . , if it satisfies the identities

B ( Av1 + pv2 , w ) = AB ( v1 , w ) + pB ( v2 , w ) and

B ( v , Aw1 + pw2 ) = AB ( v , w1 ) + pB ( v , w2 ) . %

Definition . The tensor product of the linear spaces V and W is a linear space V - - > W together with a bilinear map →: V x W ^ V - - > W , ( v , w ) ^ v ( g> w , such that for any bilinear map B: V x W ^ Z , there is a unique linear map L: V ( g> W ^ Z which makes the diagram

V x W 0 ) V <g> W

L

V x W B ) Z

commutative .

One can consider the category of bilinear maps defined on V x W , in which the objects are the bilinear maps , a morphism between the bilinear maps Bi: V x W ^ Zi and B2: V x W ^ Z2 is a linear map L: Z\ ^ Z2 , for which the diagram

x W - 1 ) Zi

L

x W — ) Z2

is commutative . In general , an object X of a category is called an initial object , if for any other object Y of the category , there is a unique morphism form X to Y . Using this terminology , the tensor product of the linear spaces V and W is the initial object of the category of bilinear maps on V x W . It is a simple exercise playing with arrows , that up to isomorphism , a category can have at most one initial object . However , initial objects do not exist in all categories . Existence of initial objects are always shown by explicit constructions in the given category .

To construct the tensor product explicitly , one first considers the linear space FV xw generated freely by the elements of V x W . More explicitly , FV xW is the linear space of all formal linear combinations Ai ( vi , wi ) + • • - + Ak ( vk , wk ) of some pairs ( vj , Wj ) G V x W with real coefficients Aj G R . Then we take the smallest linear subspace Z of FV xW that contains all elements of the form

( vi + v2 , w ) — ( vi , w ) — ( v2 , w ) , ( w , vi + v2 ) — ( w , vi ) — ( w , v2 ) , A ( v , w ) — ( Av , w ) , A ( v , w ) — ( v , Aw )

for all v , vi , v2 G V , w , wi , w2 G W , A G R . Set V O W = FVxW / Z and let O: V x W ^ V O W be the composition of the embedding V x W ^ FV xW and the factor map FV xW ^ FV xW / Z . It is not difficult to check that the bilinear map O: V x W ^ V O W is an initial object of the category of bilinear maps on V x W .

It is known that if ei , . . . , en is a basis of V , fi , . . . , fm is a basis of W , then the vectors {ej ( g> fj | 1 < i < n , 1 < j < m} form a basis of V ( g> W . In particular , dim ( V O W ) = dim ( V ) dim ( W ) .

The tensor product construction can be thought of as a functor from the category of pairs of linear spaces to the category of linear spaces In the category of pairs of linear spaces a morphism from the pair ( Vi , V2 ) to the pair ( Wi , W2 ) is a pair ( Li , L2 ) of linear maps , where Li: Vi ^ Wi and L2 : V2 ^ W2 . Tensor product as a functor assigns to a pair ( Vi , V2 ) the tensor product Vi O V2 and to the pair of linear maps ( Li , L2 ) the linear map Li O L2 : Vi O V2 ^ Wi O W2 , where the tensor product Li O L2 of the linear maps Li and L2 is defined as the unique linear map for which

Li <g> L2 ( vi <8> V2 ) = Li ( vi ) <g> L2 ( v2 ) .

Definition 1 . 2 . 21 . Let V be an n dimensional linear space and V* its dual space . The tensor product T ( k , 1 ) V = V * - - > - - - ( g> V * - - > V ^ - Cg> V will be

" V ' " V '

k times1 times

called the linear space of tensors of type ( k , l ) . We agree that T ( 0 , 0 ) V is equal to the ground field R . is

If e1 , . . . , en is a basis of V , e1 , . . . , en is its dual basis , then the type ( k , l ) tensors ejl'j = e - - > 1 - - > - - > eifc - - > ej1 - - > - - > ej form a basis of T ( k , 1 ) V . In the special case k = l = 0 the basis vector e of T ( 0 , 0 ) is the unit element of R . The direct sum ==> m= 0 t ( k , 1 ) V can be equipped with a bilinear associative tensor multiplication , which turns it into an associative algebra . Tensor product is defined on the basis vectors by the formula

e - - > 1'" ' - - > fc <S?> ePl" ' Pr = e - - > 1'" ' - - > fc P1" "Pr ejl" jl - - > eq1" ' qs ej 1'" jiq1'" qs .

If T = ' ' ' , ikj1 , '" , jl=1 Tj " " j ej1 " ' 'j is a tensor of type ( k , l ) , then the num - bers Tj1 " - - > jjl are called the coordinates or components of the tensor T with respect to the basis ei , . . . , en .

Exercise . How are the coordinates of a tensor transformed when we change the basis ei , . . . , en to another one fi , . . . , fn , where f - - > = ^n=i ajej?

Show that if ( bj ) is the inverse matrix of the matrix ( aj ) , then

nn

f - - > = b - - > ej f - - > 1" ' ifc = a - - > 1 aifc • bq1 bqfc - eP1" 'Pfc ^

1 =Z^bje , j' ' jl = aP1 . . . aPfc bj1 . . . bjfc eq1" ' ql .

Exercise . Find all t^e'l^yS ) aildrtype ( 1 , 1 ) tensors , the coordinates of which do not depend on the choice of the basis .

Exercise . The coordinates of a type ( 0 , 2 ) or a type ( 1 , 1 ) tensor can always be arranged into an n x n matrix . Do the trace and determinant of this matrix depend on the choice of the basis?

Exercise . A type ( 2 , 0 ) or a type ( 0 , 2 ) tensor is said to be non - degenerate if the n x n matrix built from its coordinates with respect to a basis has non - zero determinant . Show that non - degeneracy does not depend on the choice of the basis .

Exercise . Let <= be a non - degenerate tensor of type ( 0 , 2 ) . Show that there is a unique type ( 2 , 0 ) tensor n such that the n x n matrices built from the coordinates of <= and n with respect to any basis are inverses of one another .

Exercise . Construct natural isomorphisms between the following linear spaces:

( V C W ) * = V* C W*;

Hom ( V , W ) = V* C W , in particular , End ( V ) = T ( 1 , 1 ) V and Hom ( v , V* ) = T ( 2'0 ) V;

( T ( k , l ) V ) * = T ( k , l ) V* = T ( l , k ) V;

{V x • • • x V* x V* x • • • x V* ^ R ( k + 1 ) - linear functions} = T ( k , l ) V;

k timesl times

{V x • • • x V* ^ W k - linear maps into W} = T ( k , 0 ) V C W;

k times

{V x • • • x V* ^ V k - linear maps into V} = T ( k , 1 ) V;

k times

Hom ( T ( k'l ) V , T ( p'q ) V ) = T ( l+p . k+q ) V .

Remark . We explain what naturality of an isomorphism means for case .

For other cases a similar definition can be given . Consider the cate - gory of pairs of linear spaces , in which the objects are pairs ( V , W ) of linear spaces , the morphisms from ( V1 , W1 ) to ( V2 , W2 ) are pairs ( $ , T ) of isomor - phisms T: V1 ^ V2 and T: W1 ^ W2 . Both F1: ( V , W ) ^ Hom ( V , W ) and F2: ( V , W ) ^ V * C W are functors from this category to the category of linear spaces . If ( $ , T ) is a morphism from ( V1 , W1 ) to ( V2 , W2 ) , then the linear map F1 ( $ , T ) : Hom ( V1 , W1 ) ^ Hom ( V2 , W2 ) assigns to the lin - ear map L: V1 ^ W1 the linear map T o L o T - 1 G Hom ( V2 , W2 ) , while F2 ( $ , T ) : V* C W1 ^ V2* C W2 is the linear map ( T - 1 ) * C T . The statement that there is a natural isomorphism between Hom ( V , W ) and V* C W means that there is a natural isomorphism between the functors F1 and F2 .

There is a more practical ( but less formal ) way to characterize natural isomor - phisms . A natural isomorphism between two linear spaces is an isomorphism for which the image of an element can be described uniquely by a set of instructions or formulae . If the definition of an isomorphism involves the random choice of a basis , for example , then it may not be natural , since the image of an element may depend on the choice of the basis . On the other hand , the definition of a natural isomorphism is allowed to contain random choices , but to prove naturality , we have to check that the image of an element does not depend on the random variables .

## 9.4 EXTERIOR POWERS

Denote by Sk the group of all permutations of the set {1 , . . . , k} . Definition . Let V and W be linear spaces , k G N . A k - linear map K: Vk = V x ••• x V ^ W

N v '

k times

is said to be alternating if for any permutation a G Sk and any k vectors vi , . . . , vk G V , we have

K ( vCT ( i ) , . . . , vCT ( k ) ) = sgn a • K ( vi , . . . , Vk ) .

Alternating k - linear maps on a given linear space V form a category . A morphism from an alternating k - linear map K1: Vk ^ W1 to another one K2 : Vk ^ W2 is a linear map L: W1 ^ W2 such that K2 = L o K1 .

Definition . The kth exterior power of a linear space V is an initial object of the category of alternating k - linear maps on V . In other words , it is

**Notes**

a linear space Ak V together with an alternating k - linear map Ak : Vk ^ Ak V , ( v1 , . . . , vk ) ^ v1 A • • • A vk , such that for any alternating k - linear map K: Vk ^ W , there is a unique linear map L: AkV ^ W for which

Vk Afc > AkV

L

Vk K ) W

is commutative .

The element v1 A - ^Avk G Ak V is called the exterior product or wedge product of the vectors v1 , . . . , vk . Elements of Ak V are called k - vectors . The words bivector and trivector are also used for 2 - vectors and 3 - vectors respectively .

Is Uniqueness of the exterior power up to isomorphism follows from uniqueness of initial objects . Existence is proved by an explicit construction as follows . Consider the kth tensor power T ( 0 , k ) V and the k - linear map Ck: Vk ^ T ( 0'k ) V , ( v 1 , . . . , vk ) ^ v1 <g ) • • • <g ) vk . By the universal property of the tensor product , this is an initial object of the category of k - linear maps on V . Denote by Wk < T ( 0 , k ) V the linear hull of the set of elements of the form

v1 <g> • • • <g> vk - sgn a • vCT ( 1 ) C • • • C vCT ( k ) , where v4 G V , a G &k .

Let AkV be the factor space T ( 0 , k ) V / Wk and Ak: Vk ^ AkV be the com - position of Ck with the factor map ^k : T ( 0 , k ) V ^ T ( 0 , k ) V / Wk . It can be checked that Ak : Vk ^ Ak V is indeed an initial object of the category of alternating k - linear maps on V .

Proposition . Assume that ei , . . . , en is a basis of V . Then

{e^ A • • • A eifc | 1 < ii < ••• < ik < n} is a basis of Ak V . In particular , dimAk V = ( ^ ) .

The following formula has many applications .

Proposition . Suppose that the vectors vi , . . . , Vk can be expressed as a linear combination of the vectors wi , . . . , Wk as follows

Vi = c1 wi + + ckwk , ( i = 1 , . . . , k ) .

Then The next statement is a corollary of the previous two ones .

Proposition . The vectors vi , . . . , Vk are linearly independent if and only if Vi A • • • A Vk = 0 . Two linearly independent k - tuples of vectors Vi , . . . , Vk and Wi , . . . , Wk span the same k - dimensional linear subspace if and only if Vi A • • • A Vk = c • wi A • • • A Wk for some c G R \ {0} . In addition , these two collections are bases of the same orientation in the linear space they span if and only if c > 0 .

Corollary . The Grassmann manifold Grk ( V ) can be embedded into the projective space P ( Ak V ) by assigning to the k - dimensional subspace span - ned by the linearly independent vectors Vi , . . . , Vk the 1 - dimensional linear - space spanned by Vi A • • • A Vk . This embedding is called the Pliicker embedding .

Corollary . There is a natural one - to - one correspondence between orientations of a k - dimensional linear subspace W of an n - dimensional lin - ear space V and orientations of the 1 - dimensional linear subspace of Ak ( V ) assigned to W by the Pliicker embedding .

The direct sum A* ( V ) = 0Ak ( V ) of all the exterior powers of V be - comes an associative algebra with multiplication A defined uniquely by the rule

( Ak ( Vi , . . . , Vk ) ) A ( A; ( Vk+i , . . . , Vk+l ) ) = A k+l ( Vi , . . . , Vk+l ) Vk , l G N; vi , . . . , vk+i G V .

As we have

Ak ( vi , . . . , Vk ) = ViA . . . A Vk

for any k vectors vi , . . . , vk G V , we may ( and we shall ) denote the multi - plication A simply by A without causing confusion with the earlier notation Ak ( vi , . . . , vk ) = vi A • • • A Vk .

Definition . The associative algebra ( A* ( V ) , A ) is the exterior algebra or Grassmann algebra of V .

Exterior Powers and Alternating Tensors

Let us analyze the factor map fk : T ( 0'k ) ( V ) ^ Ak ( V ) to obtain another con - struction of the kth exterior power of V , which is , of course , naturally isomor - phic to the first construction . The permutation group Sk has a representa - tion on the tensor space T ( 0>k ) V . The representation T: Sk ^ GL ( T ( 0>k ) V ) , a i ==> - is given on decomposable tensors as follows:

( v4l <g> • • • <g> vifc ) = \a{il ) ==> • • • ==> va ( ik ) .

Definition . A tensor T of type ( 0 , k ) is called symmetric if <FCT ( T ) = T for any permutation a G Sk . T is said to be alternating if ( T ) = sgn ( a ) • T for all a G Sk . We shall use the notation Sk ( V ) C T ( 0 , k ) ( V ) for the linear space of symmetric tensors , and Ak ( V ) C T ( 0 , k ) ( V ) for the linear space of alternating tensors .

Exercise . Show that T ( ° . 2 ) ( V ) = S2 ( V ) ==> A2 ( V ) . G

Exercise . Show that dimSk ( V ) = ( n+ll - 1 ) , where n = dim V . G

Hint: Find an isomorphism between Sk ( V ) and homogeneous polynomials of degree k in n variables .

Exercise . Compute the dimension of Ak ( V ) . G

Define the linear map nk : T ( 0>k ) V ^ T ( 0>k ) ( V ) by the formula

nk ( T ) = ^ sgn ( a ) ∃CT ( T ) .

CTGSfc

Proposition . The image of the map nk is Ak ( V ) . The kernel of nk is Wk = ker fk , where fk : T ( 0>k ) V ^ T ( 0 , k ) V / Wk = Ak V is the factor map we defined in the construction of AkV . The map nk / k! is a projection onto

Ak ( V ) .

Proof . It is not difficult to check that im nk contains only alternating tensors , and if T is alternating , then nk ( T ) = k!T , so im nk = Ak ( V ) and ( nk ) / k! is a projection of T ( 0>k ) V onto Ak ( v ) .

If vi ==> • • • ==> vk — sgn a • ( vi ==> • • • ==> vk ) is a generator of Wk , then denoting vi ==> • • • ==> vk by T

nk ( T—sgn a ^ ( T ) ) ) = E sgn a' ^ ( T ) — E sgn a" •$ , , , ( T ) = 0 ,

a'E&k a / / =a / oaE&k

thus , ker nk D Wk . Conversely , if T G ker nk , then

T = k E ( T — sgn a • ( T ) ) .

ae&k

Since the image of the linear map ( I — sgn a • ) is in Wk for any a G Sk by the definition of Wk , the above expression for T shows that T is also in Wk .

According to the proposition ,

T ( 0'k ) ( V ) = Ak ( V ) ==> Wk ,

therefore , the factor space Ak ( V ) = T ( 0 , k ) ( V ) / Wk is naturally isomorphic to Ak ( V ) . There are two different ways to identify these two linear spaces . One is to identify them with the restriction Ak = Ak Afc ( V ) of the factor map

^k : t ( 0'k ) ( V ) ^A" ( V ) onto Ak ( V ) .

We can also define another natural isomorphism using the universal property of the exterior power . Since the map

nk o ==> k : Vk ^ Ak ( V ) , ( vi , . . . , vk ) ^ nk ( vi ==> • • • ==> vk )

is an alternating k - linear map , by the universal property of the exterior power , there is an induced linear map ak : Ak ( V ) ^ Ak ( V ) such that

«k ( vi A • • • A vk ) ^ nk ( vi ==> • • • ==> vk ) .

Since

Notes

Ak ( «k ( vi A • • • A vk ) = E sgn a • vCT ( i ) A • • • A vCT ( k ) = k! • vi A • • • A vk ,

CTGSfc

a - i = Ak / k! .

It is a matter of taste which isomorphism is used to identify wedge products of vectors with alternating tensors , but to avoid confusion , we should choose one of the identifications and then insist on that for the rest of the discussion . Making this decision , from now on we use ak to identify Ak ( V ) and Ak ( V ) . Nevertheless , one should be aware of the fact that some authors prefer the identification by Ak .

Taking the direct sum of the linear isomorphisms ak : Ak ( V ) ^ Ak ( V ) , we obtain a linear isomorphism a*: A* ( V ) ^ A* ( V ) between the direct sum A* ( V ) = 0Ak ( V ) and the Grassmann algebra of V . Using this iso - morphism , we can define an associative multiplication A on A* ( V ) setting Ti A T2 = a* ( a - 1 ( Ti ) A a - 1^ ) ) for Ti , T2 € A* ( V ) . More explicitly , if Ti € Ak ( V ) , T2 € A; ( V ) , then Ti = ( Ti ) / k! , T2 = n^ ) / / ! , and

T1 A T2 = ak+l ( Pk ( T1 ) A 3 ( T2 ) ) = kiTH ak+i ( ^k ( T1 ) A 3 ( T2 ) ) = ak+i ( ^k+i ( T1 G T2 ) )

k! l! ( k + l ) ! ak+i ( ^k+l ( nk+l ( T1 G T2 ) ) )

= k! ■ l! ■ ( k + l ) ! ak+i ( pk+i ( nk+i ( T1 G T2 ) ) ) = klTHnk+i ( T1 G T2 ) .

Exercise . How should we modify the formula for T1 A T2 if we used the isomorphisms f3k to identify A* ( V ) with A* ( V ) ?

Exercise . Find a natural isomorphism

Ak ( V1 ==> V2 ) - 0 Ar ( V1 ) G As ( V2 ) .

r+s = k

Exterior Powers of the Dual Space and Alternating Forms

As we saw above , there is a natural identification a*: A* ( V ) ^ A* ( V ) for any finite dimensional linear space V . Let us apply this identification to the dual space V* of V . To simplify notation , set Ak ( V ) = Ak ( V* ) , A* ( V ) = A* ( V* ) , and let ak , a* , 3k and 3* be the isomorphisms analogous to ak , a* , 3k and 3* respectively , obtained when V is replaced by the dual space V* .

Proposition . The linear space T ( k>0 ) ( V ) is naturally isomorphic to the linear space K of k - linear functions from Vk to R . Under this isomor - phism Ak ( V ) < T ( k . ° ) ( V ) corresponds to the linear space of alternating k - linear functions on V .

Proof . The first part of the statement is a special case of Exercise and can be proved .

Assign to ( 11 , . . . , 1k ) € ( V* ) k the k - linear function ^k ( 11 , . . . , 1k ) € K given by the equality

^k ( I1 , . . . , Ik ) ( v1 , . . . . , vk ) = I1 ( v1 ) ■ ■ ■ Ik ( vk ) .

Since ^k ( 11 , . . . , 1k ) depends on each Zj € V* linearly , ^k: ( V* ) k ^ K is k - linear and induces a unique linear map Tk: T ( k>0 ) V ^ K such that ^k = Tk o Ck .

We show that Tk is an isomorphism . Choose a basis ei , . . . , en of V . Then

Tk ( e* - ik ) ( eJ1 , . . . , j^ ( e^ , . . . , eik ) ( eJ1 , . . . , eJfc )

= e41 ( ej1 ) ••• eifc ( ejk ) = j ••• j ,

where <= j is the Kronecker delta symbol , thus ,

| E Ti1 . . . ik eil' . . ik I ( ej1 , . . . , ejfc ) = <= ^j - ^ ••• j = Tixj .

i1 , . . . , ik = 1 J i1 , . . . , ik = 1

Since a k linear function is uniquely determined by its values on k - tuples of basis vectors , this equality shows that the unique preimage of a k - linear function t : Vk ^ R is the tensor

n

( Tk ) - 1 ( t ) = ^ t ( ei1 , . . . , eik ) ei1 . . . ik . i1 , . . . , ik = 1

If a € Sk is a permutation , then

n

( ( *k ) - 1 ( t ) ) = <= t ( ei1 , . . . , eik ) e* ( i1 ) " - ( ik ) i1 ? . . . ?ifc —
1 n

= ^3 t ( eCT —1 ( i1 ) , . . . . , eCT —1 ( ik ) ) e 1 k ,

i1 ? . . . ?ik —1

which shows that ( Tk ) - i ( T ) is an alternating tensor if and only if t is an alternating k - linear function .

Definition . Alternating k - linear functions on a linear space V are called alternating k - forms or shortly k - forms on V . IS

The composition of the isomorphisms Tk and ak yields a natural isomorphism between Ak ( V* ) and the linear space of alternating k - forms on V . Using this natural isomorphism we shall identify the elements of the two linear spaces . If 11 , . . . , lk € V * are linear functions on V , then 11 A • • • A lk as a k - form assigns to the vectors v1 , . . . , vk € V the number

li A • • • A lk ( vi , . . . , Vk ) = ( nk ( li <g> • •• <g> lk ) ) ( vi , . . . , Vk
)

/ li ( vi ) ••• li ( vk ) N

= 53 sgna • la ( i ) ( vi ) ••• lCT ( k ) ( vk ) = det

\lk ( v i ) ••• lk ( vk )

## 9.5 EUCLIDEAN LINEAR SPACES

We know that k - linear functions on a linear space V are naturally identified with tensors of type ( k , 0 ) . A k - linear function is symmetric if and only if the corresponding tensor is symmetric , or , equivalently , if the value of the function does not change when we permute its variables .

Definition . A symmetric bilinear function ( , ) : V x V ^ R is called positive definite if ( v , v ) > 0 for all v G V and equality occurs only when v = 0 . is

Definition . A Euclidean linear space is a finite dimensional linear space V equipped with a positive definite symmetric bilinear function ( , ) , which is usually called the inner product or dot product .

For example , Rn with the standard dot product

( ( xi , . . . , Xn ) , ( yi , . . . . , Vn ) ) = Xiyi + + XnVu

on it is a Euclidean vector space .

The dot product enables us to define the length of a vector .

Definition . The length or Euclidean norm of a vector v in a Eu - clidean space V is _

IMI = . *

Proposition ( Cauchy - Schwarz Inequality ) . For any two vectors v and w in a Euclidean space V , we have

| ( vw ) | < ||v|| • ||w|| .

Equality holds if and only if v and w are linearly dependent .

Proof . If v = 0 , then both sides are equal to 0 and the vectors are linearly dependent . If v = 0 , then consider the quadratic polynomial

P ( t ) = |tv + w||2 = ||v||2t2 +2 ( v , w ) t + ||w||2 .

Since P ( t ) > 0 for all t ,

P / _ ( v , w ) \ = 2 • ( v , w ) 2 _ 2 ( v w ) • ivwl + II w|2

P v ||v||2 ; |v| i|v|4 2 ( v , w ) ||v||2 +|w|

v , w 2

= + "w"2 > 0 ,

which gives the Cauchy - Schwarz inequality after rearrangement .

P ( _iv^w )

- v + w

which implies that w = v , so v and w are linearly dependent . It is also clear that if w = Av , then both sides of the Cauchy - Schwarz inequality are equal to A||v||2 .

Corollary . For any two vectors v , w G V , we have

Ilv + w| < ||v|| + ||w|| and |||v|| - ||w|| < |v - w| .

Proof . The first inequality is equivalent to the inequality

llv + wll = llvll2 + 2 ( vw ) +llwll2 < llv|l +llw|l = M2 + 2llvIHIw|l +IM^

which follows form the Cauchy - Schwarz inequality . The second inequality is a corollary of the first one applied to the pairs ( v , w — v ) and ( w , v — w ) . □

Definition . If v and w are nonzero vectors , then there is a uniquely defined number a G [0 , n] for which

( v , w )

The number a is called the angle enclosed by the vectors v and w ( measured in radians ) .

Definition . Two non - zero vectors are orthogonal if the angle enclosed by them is n / 2 . Orthogonality of two non - zero vectors is equivalent to the condition ( v , w ) = 0 . Since the latter equation is automatically fulfilled when v = 0 , we agree , that 0 is said to be orthogonal to every vector .

Definition . A collection of some vectors ei , . . . , ek of a Euclidean linear space V is said to be an orthonormal system if ( e$ , ej ) = Aj for all 1 < i , j < k , i . e . , if the vectors have unit length and are mutually orthogonal to one another .

Theorem ( Gram - Schmidt Orthogonalization ) . Assume that the vec - tors fi , . . . , fk of a Euclidean linear space V are linearly independent then there is a unique orthonormal system of vectors ei , . . . , e^ satisfying the fol - lowing two properties:

lin[{fi , . . . , fs}] = lin[{ei , . . . , es}j for s = 1 , . . . , k;

For s = 1 , . . . , k , the ordered s - tuples of vectors ( fi , . . . , fs ) and ( ei , . . . , es ) are bases of the same orientation in the linear space they span .

Proof . According to condition , each vector es must be a linear combination

of fi , . . . , fs

ei = af , ek = ak fi + ak f2 + + akffc .

Condition on the orientation is fulfilled if and only if which means that all the diagonal elements a ss must be positive .

We prove the theorem by induction on k and give an explicit recursive formula for the computation of the vector es . For k = 1 , since e1 must be a unit vector and a 11 must be positive , the only good choice for e1 ise1

Suppose that the theorem is true for k — 1 . Then for fi , . . . , fk—i we can find an orthonormal system e1 , . . . . , ek—1 satisfying ( i ) and ( ii ) for 1 < s < k - 1 . Then ek must be of the form

ek = ( ^fcei + • • • + Pk iefc—i ) + afcfk .

Taking the dot product of both sides of ( 1 . 6 ) with ej ( 1 < i < k — 1 ) , we obtain

0 = pj + akf , ei ) ,

consequently ,

ek = ak ( fk — ( ( fk , e1 ) e1 + • • • + ( ffc , efc_i ) efc_i ^ .

The parameter ak must be used to normalize the vector which stands on the right of it . Thus ,

ak = ±llfk — ( ( fkj e1 ) e1 + • • • + ( fkj ek - 1 ) ek - i ) y 1 .

Notes

Since ak > 0 , the only possible choice of ek is

fk — ( ( fk j e1 ) e1 + • • • + ( fk j ek - 1 ) ek - 1 )

ek =

|fk - ( ( fk , e1 ) e1 + + ( fk , ek - 1 ) ek - 1 )

It is not difficult to check that the vectors e1 , . . . , ek will satisfy the requirements .

Applying the Gram - Schmidt orthogonalization to a basis we obtain the fol - lowing corollary .

Corollary Every finite dimensional Euclidean linear space has an orthonormal basis .

Definition . The Gram matrix of a system of vectors f 1 , . . . , fk of a Euclidean linear space V is the matrix

f f1 ) . . . ( f1 , fk ) N

G ( f1 , . . . j fk ) =

V ( fkjf1 ) . . . ( fk , fk ) ,

Corollary . Let G be the Gram matrix of the vectors f 1 , . . . , fk of a Euclidean linear space V . Then det G > 0 , and det G = 0 if and only if the vectors f1 , . . . , fk are linearly dependent .

Proof . If there is a non - trivial linear relation ^k=1 af = 0 , then the rows of G are linearly dependent with the same coefficients a1 therefore det G = 0 . If f1 , . . . , fk are linearly independent , then applying the Gram - Schmidt orthogonalization to them we obtain an orthonormal system e1 j . . . j ek and we can express the vectors fs as follows

f1 = a^1

fs = a^ + + a^es

e ~ 1 ~ 2 ~ k

fk = ak e1 + ak e2 + + ak ek .

The lower triangular matrix T put together from the coefficients aj is the inverse of the matrix coming from the decompositions , in particular

Corollary . With the notation used in Corollary the identity

s1

det G ( fi , . . . , fs ) = fs - J^f ei ) ei det G ( fi , . , fs - i )

i= 1

holds . Corollary . A symmetric bilinear function ( , ) on a linear space V is positive definite if and only if there is a basis fi , . . . , fn such that the Gram matrices G ( fi , . . . , fs ) have positive determinants for s = 1 , . . . , n .

Proof . The previous proposition shows that if ( , ) is positive definite , then any basis will be good . Conversely , assume that fi , . . . , fn is a

basis such that G ( fi , . . . , fs ) > 0 for s = 1 , . . . , n . Then the recursive formula

existence of an orthonormal basis implies that is positive definite , because if v = n=i viei , then ( v , v ) = ^™=i ( vi ) 2 > 0 , and equality holds only when

v = 0 .

**Check your Progress 1**

Discuss Linear Algebra

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Discuss Euclidean Linear Spaces

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## 9.6 LET US SUM UP

In this unit we have discussed the definition and example Linear Algebra , Orientation of Linear Space , Exterior Powers , Euclidean linear spaces

## 9.7 KEYWORDS

Linear Algebra … Linear Spaces and Linear Maps

Orientation of Linear Space ….A set V is a linear space , or vector space over R if V is equipped with a binary operation Let ( vi , . . . , vk ) and ( wi , . . . , wk ) be two ordered bases of a linear space V . We say that they have the same orientation or they define the same orientation of V , if the k x k matrix ( aj ) defined by the system of equalities

Exterior Powers ….. Denote by Sk the group of all permutations of the set {1 , . . . , k}

Euclidean linear spaces….. A symmetric bilinear function ( , ) : V x V ^ R is called positive definite if ( v , v ) > 0 for all v G V and equality occurs only when v = 0 . is

## 9.8 QUESTIONS FOR REVIEW

Explain Linear Algebra , Euclidean linear spaces

## 9.9 SUGGESTED READINGS

Differential Geometry, Differential Geometry & Application, Introduction to Defferential Geometry, Basic of Differential Geometry.

## 9.10 ANSWERS TO CHECK YOUR PROGRESS

Linear Algebra , Euclidean linear spaces

( answer for Check your Progress - 1 Q )

# UNIT - 10 : THE NOTION OF A CURVE

**STRUCTURE**



## 10.0 OBJECTIVES

After studying this unit , you should be able to:

- Understand about The Notion Of A Curve ,
- The Length Of A Curve ,
- Formula ,
- Crofton Formula For Spherical Curves,
- Frenet Frames And Curvatures The Fundamental Theorem Of Curve Theory

## 10.1 INTRODUCTION

Differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces

Mathematical analysis of curves and surfaces had been developed to answer some of unanswered questions that appeared in calculus like the reasons for relationships between complex shapes and curves , series and analytic functions The Notion Of A Curve , The Length Of A Curve , Crofton's Formula , Crofton Formula For Spherical Curves , Frenet Frames And Curvatures The Fundamental Theorem Of Curve Theory

## 10.2 THE NOTION OF A CURVE

In elementary geometry , one meets a lot of examples of curves: straight lines , circles , conic sections , cubic curves , graphs of functions defined on an interval or the whole real line , intersections of surfaces etc . Based on these examples everyone gets the feeling of what a curve is , however , it is not easy to give an exact definition of a curve which is satisfactory in all respect . To illustrate this , we give some commonly used definitions of certain classes of curves .

Definition A simple arc in a topological space is a subset r homeo - morphic to a closed interval [a , b] of R . A parameterization of a simple arc is a homeomorphism [a , b] ^ r . Probably anyone agrees that simple arcs are curves , but since open segments , straight lines , conic sections and many other important examples of curves are not simple arcs , this class of curves is too narrow .

We could define curves as finite unions of simple arcs . This wider class includes circles , ellipses , but still excludes non - compact examples like straight lines , hyperbolae , parabolas . Non - compact examples would be included if we considered countable unions of simple arcs . This class of curves seems to be wide enough , but maybe too wide . For example , it contains the set of all those points in Rn which have at least one rational coordinate and it is questionable whether we could call this set a curve .

Definition . A 1 - dimensional topological manifold with boundary is a second countable Hausdorff topological space , in which each point has an open neighborhood homeomorphic either to an open interval or to a left closed , right - open interval of R .

This is a technical definition , but there is a simple description of 1 - dimensional topological manifolds with boundary . They have a finite or countable number of open connected components and each connected component is homeomorphic either to an open , closed , or half - closed interval , or to a circle .

The class of 1 - dimensional manifolds with boundary is wider than the class of simple arcs , it includes much more important examples of curves , but as it fixes the local structure of a curve quite strictly , it excludes examples of curves ha / ving certain kind of singularities . For example , figure - eight shaped curves , like Bernoulli's lemniscate are not a topological manifolds , because the self - intersection point in the middle does not have a neighborhood with the required property .

Definition . An algebraic plane curve in R2 is the set of solutions of a polynomial equation P ( x , y ) = 0 , where P = 0 is a polynomial in two variables with real coefficients .

Algebraic plane curves may have a finite number of singular points , for example self intersections , so they are not necessarily 1 - dimensional manifolds , but removing the singular points , the remaining set is a 1 - dimensional manifold , maybe empty . On the other hand , algebraic curves are very specific curves . For example , if a straight line intersects an algebraic curve in an infinite number of points , then it is contained in the curve . In particular the graphs periodic non - constant functions ( like the sine function ) are not an algebraic curves .

One can also define curves as 1 - dimensional topological or metric spaces . For such a definition one must have a proper notion of dimension . Possible definitions of dimension for a topological or metric space are discussed in a branch of topology called dimension theory , and within the framework of geometric measure theory . These theories are out of the main focus of this textbook .

All the above definitions define curves as topological spaces or subsets of topological spaces having a certain property satisfied by a sufficiently large family of known examples of curves .

The second approach , which will be more suitable for our purposes , derives curves from the motion of a point . This view is reflected in the definition of a continuous curve:

Definition . A continuous parameterized curve in a topological space is a continuous map of an interval I into the space . The interval I can have any of the forms ( a , b ) ,  ( a , b] , [a , b ) , [a , b] ,  ( - to , b ) ,  ( - to , b] ,  ( a , +to ) , [a , +to ) ,  ( - to , +to ) = R , where a , b G R . If I = [a , b] , then the images of a and b are the initial and terminal points of the curve respectively . The path is said to connect the initial point to the terminal point .

We stress that according to this definition , a parameterized curve is a map and not a set of points as in the earlier definitions . However , we can associate to any parameterized curve a subset of the ambient space , the set of points traced out by the moving point .

Definition . The trace or trajectory of a parameterized curve 7 : I ^ X is the image of the map 7 . We say that 7 is a parameterization of the subset A of X when A is the trace of 7 .

In many cases , the trace of a continuous parameterized curve is a curve in one of the above sense , but there are examples , when the image is not a curve at all . The Italian mathematician Giuseppe Peano ( 1858 - 1932 ) constructed a continuous parameterized curve that passes through each point of a square . Such a pathology can not occur if we restrict ourselves to smooth curves .

Definition . A smooth parameterized curve in the Euclidean space E" is a smooth map 7: I ^ E" from an interval I into E" .     JS

Definition We say that the continuous curve 71: I1 ^ 1" is obtained from the curve 72 : I2 ^ 1" by a reparameterization if there is a homeomorphism f: I1 ^ I2 such that 71 = y2 o f . We say that the reparameterization preserves orientation if f is increasing . A reparameterization is called regular if f is smooth and f' ( t ) = 0 for all t G I1 .   Intuitively , orientation preserving reparameterizations describe motions along the same route with different timings .

## 10.3 THE LENGTH OF A CURVE

Definition . The length of a continuous curve 7: [a , b] ^ 1" is the limit of the lengths of inscribed broken lines with consecutive vertices Y ( to ) , Y ( t1 ) , . . , Y ( tN ) , where a = t0 < t1 < ••• < tN = b and the limit is taken as max1<i<N |tj — ti - 11 tends to zero . Provided that this limit is finite , the curve is called rectifiable .

Exercise . Show that the limit of the lengths of the inscribed broken lines always exists and it is equal to the supremum of the lengths . Construct a continuous curve y : [a , b] ^ 12 having infinite length . D .

The following theorem yields a formula that can be used in practice to compute the length of curves .

Theorem A smooth curve y : [a , b] ^ 1" is always rectifiable and its length is equal to the integral

l ( Y ) = f |Y' ( t ) |dt .

Proof . Denoting by x\ , x2 , . . . , xn the coordinate functions of x the length of the broken line considered in Definition 2 . 2 . 1 is equal to

N^ v ( xj xj ( ti—1 ) )

j = 1

By the Lagrange mean value theorem we can find real numbers such that

xj ( ti ) xj ( ti—1 ) xj ( Cij ) ( ti L—1 ) , ti — 1 < < tf .

Using these equalities we get

NA — El ( ti — ti— 1 )

i= 1

Fix a positive e . By Proposition 1 . 4 . 43 , for each e > 0 , we can find a positive 5 such that t , t* G [a , b] and |t — t* | < 5 imply |xj ( t ) — xj ( t* ) | < e for all 1 < j < n .

Notes

Suppose that the approximating broken line is fine enough in the sense that |tf — ti—1| <5 for all 1 < i < N . Then we have by the triangle inequality

N

Exj ( ti ) 2 ) — EE — ti—1 ) |Y / ( tij=1is just an integral sum which converges to the integral fa |y / ( t ) |dt when maxi |ti — ti—1| tends to zero . that in this case the length A of the inscribed broken lines also tends to this integral .

Let y : I ^ Rn be a smooth curve , a G I be a given point . Consider the function s : I ^ R

s ( t ) = I |y' ( t ) \dr .

J a

s ( t ) is the signed length of the arc of the curve between y ( a ) and y ( t ) . It is a monotone but not necessarily a strictly monotone function of t in general . This fact motivates the following definition .

Definition . A smooth curve is said to be regular if y' ( t ) = 0 for all t G I .

If y is a regular curve , then s defines a regular reparameterization of y . The map y ◦ s - 1: s ( I ) ^ Rn is referred to as a natural or unit speed parameterization of the curve y or as a parameterization of y by arc length . The second name is justified by the fact that the speed vector

( y ◦ s - 1»'«= Y' ( s - 1W ) • ( « - 1 ) 'W = y' ( . - 1« ) ) jy—J ) = iYy - h

of this parameterization has unit length at each point .

Exercise . The curve cycloid is the trajectory of a peripheral point of a circle that rolls along a straight line . Find a parameterization of the cycloid and compute the length of one of its arcs . D .

Sometimes it is more convenient to use the polar coordinate system in the plane .

Definition . The polar coordinates ( r , f ) of a point ( x , y ) G R2 in the plane are the distance r = \Jx2 + y2 of the point from the origin and the direction angle f of the vector ( x , y ) . The direction angle is not defined at the origin ( 0 , 0 ) , and it is defined only modulo 2n at other points . There is no continuous choice of f for the whole punctured plane R2 \ { ( 0 , 0 ) } . However , one can choose f continuously on the complement of any closed half - line starting at the origin . E . g . , on the complement of the half - line { ( x , y ) | x < 0 , y = 0} , f can be defined continuously by the formula f = 2arctan ( y I G ( — n , n ) .

Vx+v^+v v 7Cartesian coordinates can be expressed in terms of the polar coordinates as

x = r cos ( 0 ) , y = r sin ( ^ ) .

Exercise . Let 7: [a , b] ^ R2 \ {0} be a smooth curve and denote by ( r ( t ) , 0 ( t ) ) the polar coordinates of 7 ( t ) , where 0 ( t ) is chosen to be a smooth function of t . Prove that the length of 7 is equal to the integral

f Jr2 ( t ) ^'2 ( t ) + r'2 ( t ) dt .

J a

Exercise . Find a natural reparameterization of the helix 7 ( t ) = ( a cos t , a sin t , bt ) . D

## 10.4 CROFTON'S FORMULA

There are more sophisticated integral formulas for the length of a curve . In this section , we discuss some of them .

Topology and measure on the set of straight lines

Let E = AGri ( R2 ) be the set of straight lines in the Euclidean plane R2 . For ( 6 , p ) E R2 , denote by eQ , p E E the straight line defined by the equation x cos ( $ ) + y sin ( # ) = p . As every straight line can be defined by such an equation , the map p: R2 ^ E , p ( 6 , p ) = e$ , p is surjective . However , p is not injective since p ( 0 , p ) = p ( 0 , p ) if and only if 6 — 9 = kn and p = ( —1 ) kp for some integer k E Z . In particular , the band [0 , n ) x R , closed from the left and open from the right , is mapped onto E bijectively , and a boundary point ( n , p ) on its right side corresponds to the same line as the point ( 0 , —p ) . This implies easily that if we equip the set E with the factor topology induced by the surjective map p , then E will be a Mobius band without its boundary circle , and p becomes a covering map , the universal covering map of E . The Lebesgue measure A on R2 defines a measure v on E with the help of the covering map p as follows . Let a subset A C E be v - measurable if and only if p - 1 ( A ) n ( [0 , n] x R ) Lebesgue measurable and then let v ( A ) = A ( p - 1 ( A ) n ( [0 , n] x R ) ) .

Proposition . The measure v is invariant under the isometry group of the plane , that is , if T G Iso ( R2 ) is an arbitrary isometry , A C E is a v - measurable set of straight lines , then T ( A ) is also v - measurable , and v ( A ) = v ( T ( A ) ) .

Proof . The isometry group of the plane is generated by the following three types of transformations:

Rotations Ra by angle a about the origin;

Translations Ta by a vector ( a , 0 ) parallel to the x axis;

Reflection M in the x axis .

Hence it is enough to check the invariance of the measure under these transformations .

It is clear that Ra ( eg , p ) = eg+a , p . Since the translation ( 6 , p ) ^ ( 0 + a , p ) preserves Lebesgue measure , the statement is true for the rotations Ra . The action of the translation Ta on the straight line parameters is not so simple as Ta ( eg , p ) = eg , p+a cos ( 0 ) , nevertheless , as the determinant of the derivative of the transformation ( 0 , p ) ^ ( 0 , p + acos ( 0 ) ) is 1 , this transformation also preserves Lebesgue measure .

Finally , the action of M on the line parameters is given by the map ( 0 , p ) ^ ( —0 , p ) . The latter transformation is a reflection , therefore preserves the Lebesgue measure .

The Planar Crofton Formula

Theorem . Let 7: [a , b] ^ R2 is a C1 curve , m: E ^ N U {to} is defined by m ( e ) = #{t G [a , b] | 7 ( f ) G e} . In other words , m ( e ) is the number of intersection points of the curve 7 and the straight line e counted with multiplicity . Then the length of 7 is

f

lY = — m dv .

J <=

Proof . Consider the C 1 - map h: [0 , n] x [a , b] ^ [0 , n] x R defined by

h ( 0 , t ) = ( 0 , x ( t ) cos 0 + y ( t ) sin 0 ) ,

where x ( t ) and y ( t ) are the coordinates of Y ( t ) to h and the constant 1 function on [0 , n] x R . The number of preimages of a point ( 6 , p ) € [0 , n] x R is the number m ( eg , p ) of intersection points of the straight line eg , p with the curve y counted with multiplicities . The determinant of the derivative matrix of h at ( 6 , t ) is

det ( , , , . , , , „ , , , , „ 0 , , , , . „] = x' ( t ) cos 6+y' ( t ) sin 6

y—x ( t ) sm 6 + y ( t ) cos 6 x' ( t ) cos 6 + y' ( t ) sm 6 J w w

Thus , we get

/ mdv = m ( eg , p ) d6dp = / |x' ( t ) cos6 + y' ( t ) sin6|d6dt .

. / <= . P0 , nlx R a a

To compute the integral JQ |x' ( t ) cos 6 + y' ( t ) sin 6|d6 , fix the value of t and write the speed vector 7' ( t ) = ( x' ( t ) , y' ( t ) ) as v ( t ) ( cos ^> , sin ^> ) , where v ( t ) = ||Y' ( t ) || , ^ is a direction angle of the speed vector or any angle if v ( t ) = 0 . Then

pn

J <

7T

# Notes

/ |x' ( t ) cos 6 + y' ( t ) sin 6|d6 = v ( t ) / | ( cos ( 6 — ^ ) | d6 .

00

Since the absolute value of the cosine function is periodic with period n , the last integral does not depend on ^ and its value is

pn pn z»n / 2

/ | cos ( 6 — ^ ) |d6 = / | cos ( 6 ) |d6 = 2 / cos ( 6 ) d6 = 2 .

a a a

Combining these equations we get

p pb pn pb

mdv = / |x' ( t ) cos 6 + y' ( t ) sin 6|d6dt = 2v ( t ) dt = 21Y , j a jo j a

Topology and Measure on the Set of Hyperplanes

One can generalize the planar Crofton's formula for curves in Rn . In the higher dimensional version , we have to count the number of intersection points of the curve with hyperplanes . Let H = AGrn - 1 ( Rn ) denote the set of hyperplanes in Rn . Let Bn = {x € Rn | ||x|| < 1} be the unit ball centered at the origin , Sn - 1 = {x € Rn | ||x|| = 1} be its boundary sphere . Assign to each pair ( u , p ) € Sn - 1 x R the hyperplane Hu p = {x € Rn | ( u , x ) = p} . The map p: Sn - 1 x R ^ H , ( u , p ) ^ Hu p is surjective and since Hu p = Hu , p if and only if ( u , p ) = ( u , p ) or ( u , p ) = ( — u , —p ) , p is a double covering . The map p induces a factor topology and also a measure v on H . A subset A CH is v - measurable if and only if p - 1 ( A ) is ( p x A1 ) - measurable , where p is the surface measure on Sn - 1 , A1 is the Lebesgue measure on the real line , and if A is v - measurable , then its measure is v ( A ) = 1 ( p x A ) 1 ( p - 1 ( A ) ) .

Crofton Formula in Rn

Theorem 2 . 3 . 3 . Let y: [a , b] ^ Rn be a C1 curve , and m: H ^ N U {to} be the map assigning to a hyperplane H the number m ( H ) = #{t G [a , b] | Y ( t ) G H} of intersection points of the curve 7 and the hyperplane H counted with multiplicities . Then the length of 7 is mdv .

Wn - U H

where wn - 1 is the volume of the ( n — 1 ) - dimensional unit ball .

Proof . The proof is analogous to the planar case . Consider the C 1 - map

h: Bn x [a , b] ^ Bn x R , h ( x , t ) = ( x , ( x , 7 ( t ) } ) ,

and apply Theorem to it and the constant 1 function on Bn x R , For x = 0 , the number of h - preimages of ( x , p ) is the number of intersection

points of the hyperplane Hx* .

11*11 ' 11*11

The number of h - preimages of ( 0 , 0 ) is to while for p = 0 h - 1 ( 0 , p ) = 0 . However , these values can be ignored since the set {0} x R has measure 0 in Bn x R .

The determinant of the derivative matrix of h at ( x , t ) is

det ( h' ( x , t ) ) = det

= ( x , Y' ( t ) } .

Thus , Theorem yields

m ( H^* p_ ) dxdp

v ll*ll > 11*11y 1

Substituting p = ||x^pj in the second integral we see that

l ( H 1*1 , ^ ) dxdP = / ||xllm ( H |*| , p ) dxdp

R

/ / m ( Hu , p ) dudp = jRJSn - 1

n +1

To compute the integral fBn | ( x , 7' ( t ) }|dx for a fixed t , write 7' ( t ) as v ( t ) u , where v ( t ) = ||Y' ( t ) || , u is a unit vector . Slice the ball with the hyperplanes Hu , T orthogonal to u . If t G [ - 1 , 1] , then Hu , T

Notes

n Bn is an ( n — 1 ) - dimensional ball of radius a / 1 — t2 . Since the function ( x , y' ( t ) } is equal to the constant v ( t ) T on this ball ,

t 2 ) ( n - 1 ) / 2

| ( x ! ( t ) }|dx — Wn—1 v ( t ) 1 ' | ( 1 1 '

/ hu , t nBn

Integrating the integrals over the slices

1

f | ( x , 7' ( t ) }|dx — Wn—1 v ( t ) / " |t| ( 1 — t2 ) ( n 1 ) / 2dT JBn
7—1

= 2Wn—1v ( t ) f T ( 1 — T2 ) ( n—1 ) / 2dT

j 0

0

1 t 1

— 1 ( 1 — T 2 ) ( n+1 ) / 2

n +1

by Fubini's theorem . Integrating with respect to t we get

/ | ( x , y' ( t ) }|dxdt = f 2W++—11 v ( t ) dt = 2w+ - 1 / 7

JBnx[a , b] j a n + 1 n + 1

which completes the proof .

## 10.5 CROFTON FORMULA FOR SPHERICAL CURVES

To finish this section , we compute yet another version of the Crofton formula for curves lying on the unit sphere Sn — 1 C Rn . This formula will express the length of a curve using only the number of intersection points with hyperplanes passing through the origin . The scheme of producing the formula is the same as before .

Denote by H0 = Grn — 1 ( Rn ) the set of all hyperplanes passing through the origin . For u G Sn—1 , let Hu G H0 be the hyperplane orthogonal to u . The map p: Sn — 1 ^ H0 , u ^ Hu is a double cover of H0 . The spherical measure p on Sn—1 induces a measure v on H0 for which A C H0 is v - measurable if and only if p — 1 ( A ) is p - measurable and if this is the case , then v ( A ) = 2p ( A ) .

Theorem . Let 7: [a , b] ^ Sn — 1 be a spherical curve . For H G H0 , denote by m ( H ) the number of intersection points of H with 7 counted with multiplicity , i . e . ,

m ( H ) = #{t G [a , b] | y ( t ) G H} .

Then the length / Y of 7 can be obtained as the integral

2n f n f / Y =    mdv =                        m ( Hu ) du ,

nWn J H0        nWn JSn - 1

where wn is the volume of the n - dimensional unit ball .

Proof . For a fixed t G \a , b] , any vector w G Rn can be decomposed uniquely into the sum of two vectors so that the first vector is parallel to y ( t ) , the second one is orthogonal to it . The decomposition is

w = ( w Y ( t ) ) Y ( t ) + ( w -  ( w , Y ( t ) ) Y ( t ) ) .

Consider the map

h: Bn x [a , b] ^ Rn x R ,  ( w , t ) ^ ( w — ( w , Y ( t ) ) Y ( t ) ,  ( w , y ( t ) ) ) encoding the two components of this decomposition .

If ( x , s ) G Rn x R and x = 0 , then ( w , t ) G h - 1 ( x , s ) if and only if t G [a , b] is a point such that y ( t ) T w and w = x + sy ( t ) . Since ||w||2 = ||x||2 + s2 , w G Bn if and only if ( x , s ) G Bn+1 . Thus , the image of h is the unit ball Bn+1 ^ Rn x R , and the number of h - preimages of ( x , s ) G Bn+1 is

**Notes**

The determinant of the derivative matrix of h at ( w , t ) is

/ det ( h' ( w , t ) ) = det

V — ( w , Y' ( t ) ) YT ( t ) — ( w , Y ( t ) ) Y' ( t )

Adding suitable multiples of the last column to the previous ones we can eliminate the Y ( t ) YT ( t ) term from the upper left corner . Then applying we get

= ( w , Y' ( t ) ) + ( w , Y ( t ) ) ( Y ( t ) , Y' ( t ) ) .

This reduces to det ( h' ( w , t ) ) = ( w , Y' ( t ) ) as y is on the unit sphere , therefore

( llY II2 ) ' = 2 ( y , y' ) = 0 .

Applying Theorem 1 . 6 . 24 for h and the constant 1 function on Bn+1 yields

I I | ( w , Y ' ( t ) ) |dwdt =[ m ( H ) dxds

. / a JB" i ( x , s ) EB"+1 " "

Jb 2b1 — |x|2m ( HTx¥ ) dx .

The second equation comes from Fubini's theorem by slicing the ball Bn+1 by straight lines orthogonal to the hyperplane Rn x {0} . / / \ ( wr / ( t ) ) \dwdt = —— - 17 J a JBn n + 1

According to Corollary if we denote the integral f0 rn 1v / 1 — r2dr by an , then

( 2m ( H * ) V / T—Wdx = 2an f m ( Hu ) du .

JBn "X" JSn - 1

The last three equations show that

= cn / m ( Hu ) du ,

JSn — 1

where cn = ( n+1 ) an . The value of cn can be obtained by computing an , but

^n — 1

it can also be obtained by evaluating for a great circle . The length of a great circle on Sn - 1 is 2n and almost all hyperplanes through the origin cut a great circle in exactly 2 points , that is the set of exceptional hyperplanes , that contain the great circle has measure 0 . This way ,

fSn—1 m ( Hu ) du for a great circle is twice the surface measure p ( Sn - 1 ) = nw,, of the sphere . This gives that cn = i / ( nw,, ) .

Exercise . Find a direct proof of the equations

f 1

( rn 1VT—T2dr =

Jo

2k 1 ( k 1 ) ! . f n = 2 ,

( 2k + 1 ) !! , if n = 2k ,

( 2k - 1 ) !! n if n = 2k + 1 n

( 2k+2 ) !! 2 , if n = 2k + - .

## 10.6 FRENET FRAMES AND CURVATURES , THE FUNDAMENTAL THEOREM OF CURVE THEORY

Our plan is the following . A curve of general type in Rn is not contained in any affine subspace of dimension k < n — 1 ( prove this! ) , so we may pose the question how far it is from being contained in a k - plane . In other words , we want to measure the deviation of the curve from its osculating k - plane . One way to do this is that we measure how quickly the osculating flag rotates as we travel along the curve . Since the faster we travel along the curve the faster change we observe , it is natural to consider the speed of rotation of the osculating flag with respect to the unit speed parameterization of the curve . This will lead us to quantities that describe the way a curve is winding in space . These quantities will be called the curvatures of the curve . There is one question of technical character left: how can we measure the speed of rotation of an affine subspace? This problem can be solved by introducing an orthonormal basis at each point in such a way that the first k basis vectors span the osculating k - plane at the point in question , then measuring the speed of change of this basis .

Definition 2 . 5 . 1 . A ( smooth ) vector field along a curve 7: I ^ Rn is a smooth mapping v: I ^ Rn . Is Remark . There is no formal difference between a curve and a vector field along a curve . The difference is only in the interpretation . When we think of a map v: I ^ Rn as a vector field along the curve 7: I ^ Rn we represent ( depict ) v ( t ) by a directed segment starting from 7 ( t ) .

Definition . A moving ( orthonormal ) frame along a curve 7: I ^ Rn is a collection of n vector fields ti , . . . , tn along 7 such that ( tj ( t ) , tj ( t ) ) = Sj for all t € I . is

There are many moving frames along a curve and most of them have nothing to do with the geometry of the curve . This is not the case for Frenet frames .

Definition . A moving frame t1 , . . . , tn along a curve 7 is called a Frenet frame if for all k , 1 < k < n , Y ( k ) ( t ) is contained in the linear span of

ti ( t ) , . . . , tk ( t ) . ss

Exercise . Construct a curve which has no Frenet frame and one with infinitely many Frenet frames . Show that a curve of general type in Rn has exactly 2n Frenet frames . D .

According to the exercise , a Frenet frame along a curve of general type is almost unique . To select a distinguished Frenet frame from among all of them , we use orientation .

Definition . A Frenet frame t1 , . . . , tn of a curve 7 of general type in Rn is called a distinguished Frenet frame if for all k , 1 < k < n — 1 , the vectors t1 ( t ) , . . . , tk ( t ) have the same orientation in their linear span as the vectors 7 ' ( t ) , . . . , Y ( k ) ( t ) , and the basis t1 ( t ) , . . . , tn ( t ) is positively oriented with respect to the standard orientation of Rn .

Proposition . A curve of general type possesses a unique distinguished Frenet frame .

Proof . We can determine the first n — 1 vector fields of the distinguished Frenet frame by application of the Gram - Schmidt orthogonalization process pointwise to the first n — 1 derivatives of 7 . According to this recursive procedure , explained in Theorem we start with setting

t =

1 IIy 'II'

If t1 , . . . , tfc_i have already been defined , where k < n — 1 , then we compute the vector

fk = Y ( k ) — ( ( Y ( k ) , ti ) ti + + ( Y ( k ) , tfc_i ) tfc_i ) ,

and then set

, = fk = Y ( k ) — ( ( Y ( k ) , ti ) ti ++ ( Y ( k ) , tk_i ) tk_i )

k 11 fk I || Y ( k ) — ( ( Y ( k ) , ti ) ti + + ( Y ( k ) , tk_i ) tk_i ) | .

To finish the proof , we have to show that given n — 1 mutually orthogonal unit vectors ti , . . . , tn_i in Rn , there is a unique vector tn for which the vectors ti , . . . , tn form a positively oriented orthonormal basis of Rn . The condition that a vector is perpendicular to ti , . . . , tn_i is equivalent to a system of n — 1 linearly independent linear equation , the solutions of which form a 1 - dimensional linear subspace ( a straight line ) . There are exactly two opposite unit vectors parallel to a given straight line , and exactly one of them will fulfill the orientation condition . ( Replacing a vector of an ordered basis by its opposite changes the orientation . ) □

Exercise . Show that if ei , . . . , en is the standard basis of Rn and ti = Ojiei + • • • + ainen for i = 1 , . . . , n — 1 ,

then tn can be obtained as the formal determinant of the matrix

( an . . . «i,, N

a ( n_i ) i . . . a ( n_i ) n q

\ ei . . . e,, /

Proposition . Let y be a curve of general type in Rn . Denote by ti , . . . , tn its distinguished Frenet frame and set v = ||y'|| . Let the matrix ( aij ) i<i , j<n be such that

~ 1 n

- ti =Y , aij ti . ( 2 . 5 )

j=i

Then

aij = 0 provided that j > i +1 , and

the matrix ( aj ) i<i , j<n is skew - symmetric , i . e . aij = —a . ji .

Proof , ( i ) Since ti; 1 < i < n — 1 , is a linear combination of the vectors y' , . . . , y ( i ) , ti is a linear combination of the vectors y' , ••• , Y ( i+1 ) - As the latter vectors are linear combinations of the vectors ti , statement is proved .

Since ( ti , tj} = Sij is a constant function , we get

aij + aji = " ( ( ti , tj} + ( to tj} ) = 0

by differentiation . The proposition is proved .

According to Propositions only the entries ai , i+1 = — ai+1 , i of the matrix ( aij ) may differ from zero . Setting

K1 = ai2 , K2 = a23 , • • • • , Kn - 1 = a,, - i , ,, , we see that equations collapse to the following form

K1t2

V

1

— t2 — —K1t1 + K2t3

Kn—2tn—2 + Kn—1tn ^n—1cn -

These formulae are called the Frenet formulae for a curve of general type in Rn . The functions K1 , • • • , Kn - 1 are called the curvature functions of the curve .

We formulate two invariance theorems concerning the curvatures of a curve . They are intuitively clear and their proof is straightforward .

Proposition ( Invariance under isometries ) . Let y be a curve of general type in Rn , T: Rn ^ Rn be an isometry ( distance preserving bijection ) . Then the curvature functions k 1 , • • • , Kn - 2 of the curves y and T o y are the same . The last curvatures Kn - 1 of these curves coincide if T is orientation preserving and they differ ( only ) in sign if T is orientation reversing .

Proposition ( Invariance under reparameterization ) . If 7 is a regular reparameterization of the curve 7 i . e . 7 = 7 o h for some smooth function h with property h' > 0 or h' < 0 , then the curvature functions of 7 and 7 are related to one another by 7j = Kj o h for 1 < i < n — 2 and 7n - 1 = ( sgn ( h' ) ) Kn - 1 o h .

Exercise . Assume a curve 7 of general type in Rn lies in Rn - 1 C Rn . Then we can compute the curvatures k1 , . . . , Kn - 1 of this curve considering 7 a curve in Rn and also we may compute the curvatures 71 , . . . , 7n - 2 of this curve considering 7 a curve in Rn - 1 . What is the relationship between these two sets of numbers?

Computation of the Curvature Functions

Our goal now is to express the curvatures of a curve 7 of general type in Rn in terms of the derivatives of 7 . For this purpose write the derivatives of 7 as linear combinations of the distinguished Frenet frame . By the definition of the Frenet frame , the kth derivative must be the linear combination of the first k Frenet vector fields

Y' = «1t 1 ,

7" = a2t1 + a2t2 ,

( k ) ' 1 k ( 2 . 6 )

7 = ak t1 + • • • + k^^k^

( n ) 1 + 1 2 a . 1 1 nj_

7 — ant1 + ant2 + • • • + antn .

The coefficients aj can be expressed recursively by the speed length function v = 117'11 and the curvature functions . To illustrate this , let us compute the first three decompositions assuming n > 3 .

The speed vector is simply

7' = vt1 .

Differentiating and applying the first Frenet equation

7" = v't1 + vt1 = v't1 + v2K1t2 .

Differentiating again and using the first two Frenet equations we get

7"' = v"t1 + v't1 + ( v2K1 ) 't2 + v2K1t2

= v"t1 + v'vK1t2 + ( v2K1 ) 't2 + v2K1v ( K2t3 — K1t1 )

= ( v" — v3«2 ) t1 + ( v'vK1 + ( v2K1 ) ' ) t2 + v3K1K2t3 .

In principle , iterating this method we can compute all the coefficients aj but already the third derivative shows that the coefficients become complicated and do not show any pattern except for the last coefficients .

Proposition a'k = vkK1 • • • Kk—1 for 1 < k < n .

Proof . We prove the equation by induction on k . The base cases are verified above for k < 3 . Assume now that the statement holds for k — 1 < n . Then differentiating

( k —1 ) _ 1 k , , k — 2k ,

k—1^Y = ak—1t1 + • • • + ak — 1tk — 2 + ak— 1tk— 1 . and applying the Frenet formulae we obtain

Y ( k ) = Y ( ( ai—1 ) 'tj + ak—1tj )

j=1

k - 1

= ( ( ak—1 ) ' t1 + ak —1VK1t2 ) + X / ( ( afc—1 ) 'tj + afc—1V ( Kj tj+1 — Kj—1tj — 1 ) ) .

j=2

We see that tk appears only in the last summand , when j = k — 1 and its coefficient is

k k 1 / k 1 \ k i i

a - k = ak—lVKk—1 = ( v K1 ••• Kk—2 ) vKk—1 = v K1 ••• Kk—1 .

Proposition . The curvature functions K1 , . . . , Kn—2 of a curve of general type in Rn are positive . ( However , there is not any restriction on the sign of Kn—1 . )

Proof . By the orientation condition on the distinguished Frenet frame , for k < n — 1 , 7' , . . . , 7 ( k ) has the same orientation as t1 , . . . , tk in the linear space they span . According to the definition of "having the same orientation , this means that the determinant of the lower triangular matrix of coefficients is positive . Since the quotients of positive numbers is positive , we obtain also that a\ = A1 and ak = Ak / Ak—1 for 2 < k < n — 1 are positive . However , then the quotients / «k - i = VKk - i are also positive for 2 < k < n — 1 . Since v = ||y'|| > 0 , this means that all the curvatures k1 , . . . , Kn - 2 are positive .

The above computation shows also that

sgn ( Kn - i ) = sgn ( arn / arn - 1 ) = sgn ( an ) = sgn ( An / An - i ) = sgn ( An ) .

**Notes**

Consequently , the sign of the last curvature coincides with the the sign of An , which is positive ( or negative ) if and only if ( y' , . . . , Y ( n ) ) is a positively ( or negatively ) oriented basis of Rn . It is 0 if and only if y' , . . . , y ( n ) are linearly dependent . The remark at the end of the proposition follows from the Fundamental Theorem of Curve Theory , which will be proved below . This theorem that for any smooth function Kn - 1 on an interval , there is a curve of general type in Rn the last curvature of which is Kn - 1 . As a byproduct of the proof , we obtain the following expressions for the curvatures

= a2 = A2 / A 1 = A2 1 va 1 vA 1 v3

and

" k+1 _ A k+1 / A k A k+1A k - 1

K k va k vA k / A k - 1 vA2

for 2 < k < n — 1 . Thus , to find formulae for the curvatures it is enough to express the determinants Ak or the diagonal elements ak in terms of the derivatives of 7 .

If we take the wedge product of the first k equations and apply Proposition we get

y' A •••A Y ( k ) = Ak t1 A •••A tk .

As t1 , ( t ) . . . , tn ( t ) an orthonormal basis for each parameter t ,

IIy' A • • • A Y ( k ) || = |Ak| .

Since the determinants Ak are positive for k < n — 2 , the absolute value can be omitted for all but the last k . The length of the k - vector Y' A • • • A Y ( k ) can be expressed using with the help of the Gram matrix

G ( y' , . . . , Y ( k ) ) . Thus for k < n — 1 ,

Ak = IIy' A • • • A Y ( k ) I = \Jg ( y' , . . . , Y ( k ) )

Z ( y' , y' ) . . . ( Y' , Y ( k ) )

det . . . . .

V ( Y ( k ) , Y' ) . . . ( Y ( k ) , Y ( k ) ) /

We lost information on the sign of An when we took the absolute value of the sides so to obtain An together with its sign , return to equation and consider it for k = n . As the Frenet frame assigns a positively oriented orthonormal basis to each parameter , denoting by ei , . . . , en the standard basis of Rn

ti A • • • At

consequently

Y A • • • A

By the last equation we can obtain An by writing the derivatives of 7 as linear combinations of the standard basis and computing the determinant of the matrix of coefficients . If we decompose 7 as a linear combination We summarize our computation in the following theorem .

Theorem . Let y : I ^ Rn be a curve of general type in Rn , 1 < k < n — 1 . Then the curvature functions of y can be computed by the equations

Ak+iAk - i

A2 d

—and Kk K i

vAk

where v = ||y'y , and the numbers Ak are given by

When we want to compute the curvatures of a curve directly , without computing the Frenet frame , the quickest method is probably the evaluation of the formulae in the theorem . However , if we need to compute the Frenet frame as well , and we compute them following the proof of Proposition then curvatures pop up simply during the Gram - Schmidt orthogonalization process . Indeed , in the kth step of the procedure we compute the vector

fk = Y ( k ) - E<Y ( fc ) ,

i= 1

**Notes**

Taking the decompositions into account , we see that

<Y ( k ) , ti> = 4 ,

and

k - 1

fk = Y ( k )  - ^4^ = aktk .

i=1

This way , the length of fk , which we also have to compute for evaluating the equation

fk

k=

k

llfkll = 141 =vk |ki ••• Kk - i| ,

|fk+l| vll fkll

This formula allows us to compute also |Kn - 1| , but to determine the sign of Kn - 1 , one needs further consideration of the orientation of the first n derivatives of Y .

Exercise . Compute the curvatures of the moment curve

Y ( t ) = ( t , t2 ,  .  .  .  , tn ) at t = 0 .  D .

Fundamental Theorem of Curve Theory

Theorem ( Fundamental Theorem of Curve Theory ) . Given n — 1 positive smooth functions v , «q ,  .  .  . , Kn - 2 and a smooth function Kn - 1 on an interval I , there exists a curve y : I ^ Rn of general type in Rn such that ||Y'II = v and the curvatures of y are the prescribed functions Kq , .  . .  , Kn - 1 . This curve is unique up to orientation preserving isometries of the space , that is , if 7 is another curve with the same properties , then there is an orientation preserving isometry T € Iso+ ( Rn ) such that 7 = T o y .

Proof . We can eliminate the freedom given by orientation preserving isometries if we fix a parameter t0 € I and restrict our attention to curves 7: I ^Rn such that 7 ( to ) = 0 and the Frenet basis of 7 at t0 coincides with the standard basis of Rn . We are to show that within this family of curves , each collection of allowed curvature functions corresponds to a unique curve . The proof below describes a way to reconstruct the curve from its curvatures by solving some differential equations .

Let T denote the n x n matrix , whose rows are the Frenet vector fields of the unknown curve 7: I ^ Rn . If C = ( cj ) : I ^ Rnxn is the matrix valued function the only non - zero entries of which are cjji+1 = —ci+1jj = vk for 1 < i < n — 1 , then T must satisfy the linear differential equation

T' = C • T

Thus , prescribing the Frenet basis at a given point T ( t0 ) , we can obtain the whole moving Frenet frame as a unique solution For each t € I , T ( t ) must be an orthogonal matrix , that is a matrix the rows of which form an orthonormal basis of Rn . Orthogonality is equivalent to the equation T • tt = TT • T = In , where TT is the transposition of T , In is the n x n unit matrix . Furthermore , since the Frenet frame must be a positively oriented basis of Rn , T ( t ) must have positive determinant for all t € I .

Recall that orthogonal matrices with positive determinant are called special orthogonal matrices . The set ( in fact group ) of n x n special orthogonal matrices is usually denoted by SO ( n ) .

We claim that if T is a solution of there is a number to , for which T ( to ) € SO ( n ) , then T ( t ) € SO ( n ) for all t € I . This means that if we take care of the restrictions on T when we choose the initial matrix T0 , then we do not have to worry about the other values of T .

To show the statement , set M = TT • T . Then

M' = ( T' ) T • T + TT • T' = ( C • T ) T • T + TT • C • T = TT • ( CT + C ) • T = 0 ,

Notes

as C is skew symmetric shows that M is constant , therefore M = M ( t0 ) = In . This proves that T ( t ) is an orthogonal matrix for all t € I . As for the determinant of T ( t ) , observe that the determinant of an orthogonal matrix A € Rnxn satisfies

( det A ) 2 = det A • det AT = det ( A • AT ) = det In = 1 ,

hence det A = ±1 . Since det ( T ( t ) ) is a continuous function of t , which takes only the values ±1 , it must be constant . We have assumed that T ( t0 ) > 0 , thus T ( t ) = 1 > 0 for all t € I .

The derivative of 7 is vti , thus we can obtain 7 from the Frenet frame by integrating vti defines the only curve which can satisfy the conditions . This proves uniqueness . To show the existence part of the theorem we have to check that the curve we have obtained has the prescribed curvature and length of speed functions .

Exercise Plot the curve , called astroid , given by the parameterization 7: [0 , 2n] ^ R2 , 7 ( f ) = ( cos3 ( t ) , sin3 ( t ) ) . Is the astroid a smooth curve? Is it regular? Is it a curve of general type? If the answer is no for a property , characterize those arcs of the astroid which have the property . Compute the length of the astroid . Show that the segment of a tangent lying between the axis intercepts has the same length for all tangents .

Exercise . Find the distinguished Frenet frame and the equation of the osculating 2 - plane of the elliptical helix t ^ ( a cos t , b sin t , ct ) at the point ( a , 0 , 0 ) ( a , b and c are given positive numbers ) .

Exercise . Show that a curve of general type in Rn is contained in an ( n — 1 ) - dimensional affine subspace if and only if k,,_i = 0 .

Exercise . Prove that if a curve of general type in R3 has constant curvatures then it is either a circle or a helix .

Exercise Describe those curves of general type in Rn which have constant curvatures .

Obtaining Curvatures

as Limits of Some Geometrical Quantities

In this section , we compute the limit of some geometrical quantities depending on a collection of points on the curve as the points tend simultaneously to a given curve point . The limits will be expressed by the curvatures at the given point , thus , each of these results gives rise to a geometrical interpretation of curvatures .

**Check your Progress 1**

Discuss Notion Of A Curve

_______________________________________________________

_______________________________________________________

_______________________________________________________

Discuss Crofton Formula For Spherical Curves

______________________________________________________

_______________________________________________________

_______________________________________________________

## 10.6 LET US SUM UP

In this unit we have discussed the definition and example The Notion Of A Curve , The Length Of A Curve , Crofton's Formula , Crofton Formula For Spherical Curves , Frenet Frames And Curvatures The Fundamental Theorem Of Curve Theory

## 10.7 KEYWORDS

The Notion Of A Curve …. In elementary geometry , one meets a lot of examples of curves: straight lines , circles , conic sections , cubic curves , graphs of functions defined on an interval or the whole real line , intersections of surfaces etc

The Length Of A Curve ….. The length of a continuous curve 7: [a , b] ^ 1" is the limit of the lengths of inscribed broken lines with consecutive vertices Y ( to ) , Y ( t1 ) , . . . , Y ( tN ) , where a = t0 < t1 < ••• < tN = b and the limit is taken as max1<i<N |tj — ti - 11 tends to zero

Crofton Formula ….. Topology and measure on the set of straight lines

Crofton Formula For Spherical Curves … This formula will express the length of a curve using only the number of intersection points with hyperplanes passing through the origin

Frenet Frames And Curvatures The Fundamental Theorem Of Curve Theory … A curve of general type in Rn is not contained in any affine subspace of dimension,

## 10.8 QUESTIONS FOR REVIEW

Explain The Notion Of A Curve , Crofton Formula For Spherical Curves

## 10.9 SUGGESTED READINGS

Differential Geometry, Differential Geometry & Application, Introduction to Defferential Geometry, Basic of Differential Geometry.

## 10.10 ANSWERS TO CHECK YOUR PROGRESS

The Notion Of A Curve , Crofton Formula For Spherical Curves

( answer for Check your Progress - 1 Q )

# UNIT-11 : PLANE CURVES

**STRUCTURE**



## 11.0 OBJECTIVES

After studying this unit , you should be able to:

- Understand about Plane Curves
- Convex Curves
- Interior Product By A Vector Field
- Exterior Differentiation

## 11.1 INTRODUCTION

Differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces Mathematical analysis of curves and surfaces had been developed to answer some of unanswered questions that appeared in calculus like the reasons for relationships between Plane Curves , Convex Curves , Interior Product By A Vector Field , Exterior Differentiation

## 11.2 PLANE CURVES

This section we deduce some facts on plane curves from the general theory of curves . A plane curve 7: I ^ $R^2$ is given by two coordinate functions .

Y ( t ) = ( x ( t ) , y ( t ) ) t e I .

The curve 7 is of general type if the vector 7' is a linearly independent "system of vectors" . Since a single vector is linearly independent if and only if it is non - zero , curves of general type in the plane are the same as regular curves . From this point on we assume that 7 is regular .

The Frenet vector fields $\mathbf{t_1}$ **,** $\mathbf{t_2}$ are denoted by **t** and **n** in classical differential geometry and they are called the ( unit ) tangent and the ( unit ) normal vector - fields of the curve . There is only one curvature function of a plane curve k = $K_1$ . The Frenet formulae have the form

t' **= vK**n , n' **= —vK**t ,

where v = |Y'| .

Let us find explicit formulae for **t , n** and K . Obviously

The normal vector **n** is the last vector of the Frenet basis so it is determined by the condition that **( t , n )** is a positively oriented orthonormal basis , that is , in our case , **n** is obtained from **t** by a 90 degree rotation in positive direction .

The right angled rotation in the positive direction takes the vector ( a , b ) to the vector ( —b , a ) , thus

**n** = 1 ( —y' , x' ) = ~~.2~~ = ( —y' , x' ) . v Vx'$^2$ + y'$^2$

To express K , let us start from the equation

Y' = v**t .**

Differentiating and using the first Frenet formula ,

Y'' = v'**t** + v**t'** = v'**t** + $v^2$K**n .**

Taking dot product with **n** and using **( t , n )** = 0 we get

( y'' , **N )** = $v^2$k ,

which gives

x' y'

$$( \gamma'' , \mathbf{n} ) —x''y' + y''x' {}^{de}\backslash x'' y''$$

$V^2$ $V^3$ $( x'^2 + y'^2 )^{3/2}$ '

in accordance with Theorem

Evolute , Involute , Parallel Curves

Since a 1 - sphere in a plane is simply a circle , osculating 1 - spheres of curve are called its osculating circles . The osculating circle of a regular plane curve Y exists at a point t if and only if K ( t ) = 0 . Then the center **o**i ( t ) of the radius $R_1$ ( t ) of the osculating circle are given by

$$\mathbf{o}_i ( t ) = Y ( t ) + - \wedge \mathbf{n} ( t ) , R_i ( t )^{1}$$

**Definition .** The center of the osculating circle is called the center of curvature , the radius of the osculating circle is called the radius of curvature of the given curve at the given point .

The center of the osculating circle can be obtained also as the limit of the intersection point of two normal lines of Y .

**Theorem .** Assume that Y : I ^ $R^2$ is a smooth regular plane curve , t € I is such that K ( t ) = 0 . Denote by M ( to , ti ) the intersection point of the normals of Y taken at to and ti = to . Then M ( to , ti ) exists if to and ti are sufficiently close to t and

$$\lim M ( t_o , t_i ) = \mathbf{o}_i ( t ) .$$

$$t_{\mathbf{o}} , t_{\mathbf{i}} —\mathbf{yt}$$

Proof . The intersection point of the normals exist if the corresponding tan - gents are not parallel . Since **t'** ( t ) = « ( t **)** **n** ( t ) = **0 ,** the unit tangent **t** rotates with nonzero angular speed at t , hence no two tangents are parallel in a small neighborhood of t .

Assume $t_o$ , ti are close to t and write M ( $t_0$ , t\ ) as

$$M ( to , ti ) = Y ( to ) + m ( to , t_1 ) \mathbf{n} ( to )$$

Notes

using the fact that $M(t_0, t_1)$ is on the normal at $t_0$. Since $M(t_0, t_1)$ is also on the normal at ti ,

( M ( to , ti ) - Y ( ti ) , **t** ( ti ) ) = ( Y ( to ) + m ( to , ti **)** **n** ( to ) - Y ( ti ) , **t** ( ti ) ) = 0 .

Solving this equation for $m(t_o, t_i)$ gives

Y - Y ( to ) , **t« . )**

( t t ) ( Y ( ti ) - Y ( to ) , t ( ti ) ) \ ti - to m (
to , ti ) =

( n ( to ) , t ( ti ) ) / n ( to ) - n ( ti ) t ( ^ )

ti - to

It is easy to calculate the limit of the right - hand side

m ( t t ) **t** ( t ) ) **t** ( t ) )

lim m ( $t_o$ , tl ) = —;— . $t_0$ , $t_1$**—t** **- ( n'** ( t ) , **t** ( i ) ) (
v ( t ) K ( t **)** **t** ( t ) , **t** ( i ) ) K ( t )

In conclusion ,

lim M ( to , ti ) = lim Y ( to ) + m ( to , ti **)** **n** ( to ) = Y ( t ) + ^w"**n** ( t ) = **o**i ( t ) . □

t**o** , t**i**——t to , t**i**—yt K ( t )

**Exercise .** Let us illuminate a curve by rays of light , parallel to the normal line of Y at Y ( t ) . Denote by F ( $t_i$ , $t_2$ ) the intersection point of the rays reflected from Y ( ti ) and Y ( $t_2$ ) . Show that $\lim_{t_i t_2 \to t}$ F ( $t_i$ , $t_2$ ) ^ Y ( t ) +2KY ) t2 ( t ) '

**Definition .** The locus of the centers of curvature of a curve is the evolute of the curve . The evolute is defined for arcs along which the curvature is not zero .

If Y : I ^ $R^2$ is a curve with a nowhere zero curvature function K and unit normal vector field **n ,** then the evolute can be parameterized by the mapping $\mathbf{o_i}$: I ^ $R^2$ , $\mathbf{o_i}$ = Y + ( 1 / **« ) n .**

**Exercise .** Show that the evolute of the ellipse Y ( t ) = ( a cos t , b sin t ) is the "affine astroid" **o**$_i$ ( t ) = ^ọ2 – &2 cos$^3$1 , b2 – ọ2 sin$^3$ tj .

The evolute of a curve was introduced by Christiaan Huygens ( 1629 - 1695 ) . He used the fact that the evolute of a cycloid is a congruent cycloid to modify the pendulum used in pendulum clocks to increase the accuracy of the clock . He could also be led to the notion of the evolute in connection with his investigations on the propagation of wave fronts . If we generate a curvilinear wave on the surface of calm water ( e . g . we drop a wire into it ) , the wave starts moving . Geometrically , the shape of the wave front at consecutive moments of time is described by the parallel curves of the original curve .

**Definition .** Let 7 be a regular plane curve with normal vector field **n .** A parallel curve of 7 is a curve of the form $y_d$ = ^ + d**n ,** where d G R is a fixed real number .

Experiments on wave fronts show that even if the initial curve is smooth and regular , singularities may appear on its parallel curves . As the distance d varies , singularities on are usually born and disappear in pairs .

**Definition .** Let Y be a smooth parameterized curve , t a point of its parameter interval . We say that 7 ( t ) ( or t ) is a singular point ( or singular parameter ) of the curve 7 if 7' ( t ) =0 .

**Exercise .** Study the singularities of the parallel curves of an ellipse .

**Proposition .** Singular points of the parallel curves of a regular curve Y trace out the evolute of 7 .

Proof . Since $y'_d$ = 7' + d**a!** = v**t** — vdK**t** = ( 1 — dK ) v**t ,** singular parameters are characterized by 1 — dK ( t ) = 0 . Then the corresponding singular points Yd ( t ) = Y ( t ) + ( 1 / K ( t **) ) n** ( t ) lie on the evolute of the curve . It is also easy to show that any evolute point is a singular point of a suitable parallel curve .

**Proposition** Let 7: [a , b] ^ R$^2$ be a regular plane curve with mono - tone non - vanishing curvature K , and evolute **o**i = 7 + ( 1 / K **) n .**

- If K' ( t ) = 0 , then **o**i is regular at t and its tangent line at t coincides with the normal line of 7 at t .

- The length of the arc of the evolute between **o**i ( ti ) and **o**i ( t2 ) , ti < t**2** **,** is the difference of the radii of curvatures 11 / K ( 7 1 ) — 1 / K ( t2 ) | .

Proof . The speed vector of **o**i is

$\mathbf{o_1}' = Y' + (1/K)'\mathbf{n} + (1/K)\mathbf{n}^d = v\mathbf{t} + (1/K)'\mathbf{n} - (1/K)vK\mathbf{t} = (1/K)'\mathbf{n}$.

This equation shows that if $K^d(t) = 0$, then $\mathbf{o_1}$ is regular at t and its tangent at t is parallel to the normal **n** ( t ) of Y . This proves the first part of the proposition . As for the length of the evolute , it is equal to the integral

F |OI' ( R ) \DR = F | ( 1 / k ) ' ( t ) N ( R ) \DR = F | ( 1 / k ) ' ( t ) \DR ITL JT 1 JT 1

**f**2

( 1 / K ) , ( T ) dT t**1**

At the third equality we used that 1 / K is monotone , thus either ( 1 / K ) ' > 0 or ( 1 / K ) ' < 0 everywhere .

The above proposition gives a method to reconstruct the curve 7 from its evolute . Suppose for simplicity that K > 0 and K' > 0 . Take a thread of length 1 / k ( o ) and fix one of its ends to $\mathbf{o}_1$ ( a ) . Then pulling the other end of the thread from the initial position 7 ( a ) wrap the thread tightly onto the curve $\mathbf{o_1}$ .

**Definition .** Let 7: I ^ $R^2$ be a regular curve with unit tangent vector field **t ,** a G I . An involute of the curve 7 is a curve 7 of the form Y = 7 + ( l — s **) t ,** where l is a given real number , s ( t ) = $f_a$ |Y' ( T ) |dT is the signed length of the arc of 7 between 7 ( a ) and 7 ( t ) .

A curve has many involutes corresponding to the different choices of the initial point a G I and the length l of the thread .

**Proposition .** Let 7 be a unit speed curve with K > 0 , 7 ( s ) = 7 ( s ) + ( l — s **) t** ( s ) an involute of it such that I is greater than the length of 7 . Then the evolute of 77 is 7 .

Proof . We have

$$Y'(s) = \mathbf{t}(s) - \mathbf{t}(s) + (1-s)\kappa(s)\mathbf{n}(s) = (1-s)\kappa(s)\mathbf{n}(s),$$

$$7''(s) = [(1-s)K(s)]'\mathbf{n}(s) - (1-s)K^2(s)\mathbf{t}(s).$$

The first equation implies that the Frenet frame **t , n** of 7 is related to that of Y by **t** = **n , n** = — **t .** Computing the curvature 7 of 7 ,

$$= \frac{(7''(s), \mathbf{n}(s))}{|Y'(s)|^3} = \frac{(1-S)^2K^3(S)}{(1-s)^3K^3(s)} = \frac{1}{(1-s)}$$

Thus , the evolute of Y is $7 + (1/7)\mathbf{n} = Y + (1-s)\mathbf{t} - (1-s)\mathbf{t} = Y$.

We formulate some further results on the evolute and involute as exercises .

**Exercise** Suppose that the regular curves YI and Y2 have regular evolutes . Show that YI and $Y_2$ are parallel if and only if their evolutes are the same .

Show that if two involutes of a regular curve are regular , then they are parallel .

**Exercise** The curve cardioid is the trajectory of a peripheral point of a circle rolling about a fixed circle of the same radius .

- Find a smooth parameterization of the cardioid .
- Compute its length .
- Show that its evolute is also a cardioid .

- Determine the involute of the chain curve through the point ( 0 , 1 ) . ( This curve is called "tractrix" . )
- Let the tangent of the tractrix at P intersect the x - axis at Q . Show that the segment PQ has unit length .

# 11.3 CONVEX CURVES

**Definition .** A simple closed plane curve 7 is convex , if for any point **p** = Y ( t ) , the curve lies on one side of the tangent line of 7 at **p .** In other words the function ( 7 ( t ) — **p , n** ( t ) } must be > 0 or < 0 for all t .

**Exercise .** Show that a simple closed curve is convex if and only if every arc of the curve lies on one side of the straight line through the endpoints of the arc .

Convex curves can be characterized with the help of the curvature function .

**Theorem .** A simple closed curve is convex if and only if κ > 0 or κ < 0 everywhere along the curve .

Proof . Assume first that 7 is a naturally parameterized convex curve . Let a ( t ) be a continuous direction angle for the tangent **t** ( t ) . As we know , a' = κ , thus , it suffices to show that a is a weakly monotone function . This follows if we show that if a takes the same value at two different parameters ti , $t_2$ , then a is constant on the interval [ti , $t_2$j . The rotation number of a simple curve is ±1 , hence the image of **t** covers the whole unit circle . As a consequence , we can find a point at which

$$t(t_3) = -t(t_1) = -t(t_2).$$

If the tangent lines at $t_1$ , $t_2$ , $t_3$ were different , then one of them would be between the others and this tangent would have points of the curve on both sides . This contradicts convexity , hence two of these tangents say the tangents at **p** = y ( tj ) and **q** = 7 ( tj ) coincide .

We claim that the segment **pq** is an arc of 7 . It is enough to prove that this segment is in the image of 7 . Assume to the contrary that a point **x** G **pq** is not covered by 7 . Drawing a line e = **pq** through **x ,** we can find at least two intersection points **r** and **s** of e and the curve , since e separates **p** and **q** and 7 has two essentially disjoint arcs connecting **p** to **q .** Since **pq** is a tangent of 7 , the points **r** and **s** must lie on the same side of it . As a consequence , we get that one of the triangles **pqr** and **pqs ,** say the first one is inside the other . However , this leads to a contradiction , since for such a configuration the tangent through **r** necessarily separates two vertices of the triangle **pqs ,** which lie on the curve .

If Y is defined on the interval [a , b] , then 7 ( a ) = 7 ( b ) is either on the segment **pq** or not . The first case is not possible , because then a would be constant on the intervals \a , t\] and [$t_2$ , b] , yielding

a ( a ) = a ( ti ) = a ( $t_2$ ) = a ( b )

and

rotation number = ( a ( b ) — a ( a ) ) / 2n = 0 .

In the second case a is constant on the interval [ti , $t_2$] , as we wanted to show . Now we prove the converse . Assume that 7 is a simple closed curve with k > 0 everywhere and assume to the contrary that 7 is not convex ( the case k < 0 can be treated analogously ) . Then we can find a point **p** = 7 ( $t_1$ ) , such that the tangent at **p** has curve points on both of its sides . Let us find on each side a curve point , say **q** = Y ( $t_2$ ) and **r** = Y ( $t_3$ ) respectively , lying at maximal distance from the tangent at **p .** Then the tangents at **p , q** and **r** are different and parallel . Since the unit tangent vectors **t** ( tj ) , i = 1 , 2 , 3 have parallel directions , two of them , say **t** ( tj ) and **t** ( tj ) must be equal . The points **a** = 7 ( tj ) and **b** = q ( tj ) divide the curve into two arcs . Denoting by $K_1$ and $K_2$ the total curvatures of these arcs , we deduce that these total curvatures have the form $K_1$ = $2k_1n$ , $K_2$ = $2k_2n$ , where $k_1$ , $k_2$ G **Z ,** since the unit tangents at the ends of the arcs are equal . On the other hand , we have k1 + k2 = 1 by the Umlaufsatz and k1 > 0 , k2 > 0 by the assumption K > 0 . This is possible only if one of the total curvatures $K_1$ or $K_2$ is equal to zero . Since K > 0 , this means that K = 0 along one of the arcs between **a** and **b .** But then this arc would be a straight line segment , implying that the tangents at **a** and **b** coincide . The contradiction proves the theorem .

Differential Forms

A differential k - form w on a manifold M assigns to each point p G M an element of Ak ( TpM ) * . As elements of AkV* for a linear space V can be identified with the linear space Ak ( V ) of alternating k - linear functions , we can evaluate w ( p ) on any k tangent vectors vi , . . . , vk

G TpM . In particular , if we are given k smooth vector fields X1 , . . . , Xk on M , then we can evaluate w on X1 , . . . , Xk pointwise and get a smooth function on M . This function will be denoted by w ( X1 , . . . , Xk ) .

The pointwise wedge product of a differential k - form w G 0k ( M ) and a differential l - form n G Ql ( M ) is a differential ( k + 1 ) - form . Wedge product of differential forms is not a commutative operation . Instead of that it satisfies the super commutativity relation

w a n = ( —i ) kln a w .

## 11.4 INTERIOR PRODUCT BY A VECTOR FIELD

Definition . The interior product of a smooth vector field X and a differential k - form w on a differentiable manifold M is a differential ( k — 1 ) - form iX w . It is defined by the property that

ixw ( Xi , . .  . , Xk - i ) = w ( X , Xi , . . . , Xk - i ) .

When k = 0 , we set iXw = 0 .            IS

Interior product with the vector field X can be thought of as a linear map from Q* ( M ) = 0^k ( M ) into itself . This is a degree - 1 map , as it decreases the degree of any form by 1 .

Definition 4 . 6 . 2 . In general , a linear map L: H* ( M ) ^ H* ( M ) is a degree d map , if L ( Qk ( M ) ) C Qk+d ( M ) for all k .       IS

It is interesting that although interior product has a linear algebraic character , it satisfies a Leibniz - type rule .

Definition . A linear map L: H* ( M ) ^ H* ( M ) is called a superderivation if for any w G Qk ( M ) and n G Ol ( M ) , we have

L ( w a n ) = L ( w ) a n + ( —i ) k w a L ( n ) .

Proposition . Interior product with the vector field X is a super derivation .

Proof . Both sides are bilinear functions of w and n , therefore , it is enough to check the graded version of the Leibniz rule for forms that generate Qk ( M ) and Ql ( M ) additively . It is also enough to show the graded Leibniz identity on domains of charts , since any point is contained in the domain of a chart . In the domain of a chart , any k - form can be written as the sum of wedge products of k - tuples of 1 - forms , so it is enough to show the graded Leibniz rule for the special case when both w and n are wedge products of 1 - forms . Let li , . . . . , ls be 1 - forms . Then

/ li ( X ) li ( Xi ) . . . l i ( Xs_i ) \ l2 ( X ) l2 ( Xi ) . . . l2 ( Xs - i )

ix ( li A • • • A ls ) ( Xi , . . . , Xs - i ) — det

\ls ( X ) ls ( Xi ) . . . ls ( Xs - i ) /

Computing the determinant by Laplace expansion with respect to the first column we obtain

ix ( li A • • • A ls ) ( Xi , . . . , Xs - i )

— e ( - 1 ) i+ili ( X ) • ( li A • • • A l A • • • A ls ) ( Xi , . . . , Xs - i ) ,

i=i

where the hat over li means that it is omitted . This means that

ix ( li A ••• A ls ) — ^ ( - 1 ) i+ili ( X ) • ( li A ••• A li A ■■■ A ls ) .

i=i

Consider the special case of the statement , when w — li A • • • A lk and n — lk+i A • • • A lk+i . Then

k+l

ix ( w A n ) — ix ( li A • • • A lk+i ) — e ( - 1 ) i+ili ( X ) • ( li A • • • A li A • • • A lk+l )

i=i

k

— ^3 ( —1 ) i+ili ( X ) • ( li A • • • A li A • • • A lk+l ) +

+ ^"~1 ( — 1 ) k+i+ilk+i ( X ) • ( li A • • • A lk+i A • • • A lk+l )

i=i

— ix ( w ) a n + ( - 1 ) kw a ix ( n ) .

By the introductory arguments , this special case implies the general one . Exercise . Show that for any two vector fields X an Y , we have

ix ° iY — — iY ° ix .

Exercise 4 . 6 . 6 . Show that the Lie derivation of differential forms and the in - terior product with a vector field satisfies the commutation relation

Lx ◦ lY - lY ◦ Lx = l[X , Y] •

## 11.5 EXTERIOR DIFFERENTIATION

We can associate to each smooth function a differential 1 - form , its differential .

Definition . The differential df of a smooth function f G QP ( M ) is the differential 1 - form df G H1 ( M ) , the evaluation of which on a vector field X is given by df ( X ) = X ( f ) .

Remark . If $ = ( x1 , . . . , xn ) : U ^ Rn is a chart , then the 1 - forms determined by the pointwise dual basis of the basis vector fields df , . . . , df were denoted by dx1 , . . . , dxn . The notation is motivated by the fact that these 1 - forms can also be obtained as the differentials of the coordinate functions .

Indeed ,

dxi ( df ) = df ( ff ) = j .

Our goal is to extend the differential d from functions to all differential forms .

Theorem There is a unique superderivation d: Q* ( M ) ^ Q* ( M ) of degree 1 , which has the following properties .

For f G Q° ( M ) , df is the differential of the function f .

d o d = 0 .

The superderivation d defined uniquely by the theorem is called the exterior differentiation of differential forms .

Proof . To prove uniqueness , first observe that if two k forms a1 , a1 coincide on an open neighborhood V of a point p , then dw1 and dw2 must coincide at p . Indeed , choose a smooth bump function h G QP ( M ) such that supp h C V and h = 1 on a neighborhood of p . Then h ( a1 — a2 ) =0 , therefore ,

0 = d ( h ( a 1 — a2 ) ) = dh A ( ^1 — a2 ) + h ( dw1 — da2 ) .

Evaluating at p gives ( da1 ) ( p ) = ( da1 ) ( p ) .

Let p be an arbitrary point in M and choose a chart $ = ( x1 , . . . , xn ) : U ^ Rn around p . Fix also a smooth function g G QP ( M ) such that supp g C U and g|V = 1 on an open neighborhood V C U of p . Introduce the notation that for a smooth function f defined on U , denote by f the smooth function on M defined by

7 ( ) = Jf ( q ) g ( q ) , if q g u ,

f ( q ) = 0 , otherwise .

Let w be an arbitrary differential k - form . We can write w| U uniquely as

w|U wi1 . . ik dx11 A •••A dxlk .

1<ii <••• <ik <n

Consider the differential form w G ( M ) defined by

w = wil . . . ik dx11 A •••A dxik .

1<ii <••• <ik <n

The forms w and w coincide on V and the computation of d on w is uniquely dictated by the properties of d , thus ,

( dw ) ( p ) = ( dw ) ( p ) = i e dwil . . . ik A dx11 A • • • A dxik i ( p ) .

\l<ii < - <ik<n I

Since by this method we can compute the value of dw at any point p , d is uniquely defined by its properties .

For the existence , we first construct for each chart y = ( x1 , . . . , xn ) : U ^ Rn an exterior differentiation d: Q* ( U ) ^ Q* ( U ) on U . For w G Q* ( U ) write w as

w wil . . . ik dx11 A •••A dxik ,

1<ii <••• <ik <n

and set

d^w dwi1 . . . ik A dx11 A ••• A dxik .

1<ii <••• <ik <n

Let us check that d^ satisfies all the requirements . It is obviously linear and increases the degree of forms by 1 , and coincides with d on smooth functions . It is also a super derivation . Indeed , if n G Ql ( M ) has the decomposition

n Vj1 . . . ji dxj1 A ••• A dxjl ,

1<ji< - <jl <n

then

d^ ( w A n ) =

d^ ( ^3 ^3 wi1 . . . ik • nj1 . . . jl dxi1 A • • • A dxik A dxj1 A • • • A dxjl^j

1<ii< - <ik<n 1<ji< - <jl<n

= d ( wi1 . . . ik • nj1 . . . jl ) A dx11 A ••• A dxik A dxj1 A ••• A dxjl

1 <i 1 < •• <ik <n 1<ji < • •• <jl <n

For smooth function f and g , the Leibniz - type formula d ( f • g ) = df • g + f • dg holds , since for any smooth vector field X , d ( f • g ) ( X ) = X ( f • g ) = X ( f ) • g + f • X ( g ) = ( df • g + f • dg ) ( X ) .

Applying this special case to the general one , we obtain

d$ ( w A n ) =

e e nj1 . . . jl • dwi1 . . . ik A dx11 A ••• A d , xlk A dxj1 A ••• A dxjl

i<il< - <ifc <n 1<j1< - <ji<n

+ e e wi1 . . . ik • dnj1 . . . jl A dx11 A • • • A dxik A dxj1 A • • • A dxjl

1<i1< - <ifc <n 1<j1< - <ji<n

= dw A n + ( —1 ) kw A dr / .

It remains to show that d$ o d$ = 0 . As d$ is a superderivation , d$ ( d$ ( w ) ) =

( d$ ( dwi1 . . . ik ) A dx ==> 1 A • • • A dxik — dwi1 . . . ik A d$ ( dxi1 A • • • A d , xlk ) )

1<i1 <••• <ik <n

= d$ ( dwi1 . . . ik ) A dx11 A •••A dxik ,

1<i1 <••• <ik <n

it is enough to prove that d$ ( d ( f ) ) = 0 for any smooth function f . Since dx1 , . . . , dxn is the pointwise dual basis of d$ , . . . , d$ , df can be decomposed as a linear combination of the 1 - forms dxi as

n

df = e ( d i$f ) dxi .

i=1

From this formula ,

n n n

d$ ( df ) = e d ( d$f ) A dxi = e E ( d$d$f ) dxj A dxi

i= 1 i= 1 j = 1

= ( d$d$f — d$d$f ) dxi A dxj = 0 .

1< i<j< n

If we consider two charts , $: U ^ Rn and ^: V ^ Rn , then the restrictions of both $ and ^ onto U fl V give an exterior differentiation on U n V . But the uniqueness part of the theorem implies that these exterior derivations must be the same . This means that if w is a globally defined k - form on M , then the forms d$ ( w|U ) taken for all charts $: U ^ Rn glue together to a global ( k + 1 ) - form dw on M . The map w ^ dw is an exterior derivation , since all the properties can be checked locally , using a local coordinate system .

In the rest of this section we prove some useful formulae involving the exterior differentiation .

Proposition . If f: M ^ N is a smooth map , w G Uk ( N ) is a differ - ential k - form on N , then d ( f * ( w ) ) = f * ( dw ) .

Proof . The statement is true for smooth functions h G QP ( M ) , because for any smooth vector field X on M and for any p G M , we have

d ( f * ( h ) ) ( Xp ) = Xp ( f * ( h ) ) = Xp ( h o f ) = ( Tpf ( Xp ) ) ( h )

= dh ( Tpf ( Xp ) ) = ( f * ( dh ) ) ( Xp ) .

It is also clear that a wedge product can be pulled back component wise , i . e . , f * ( w A n ) = f * ( w ) A f * ( n ) .

If p G M is an arbitrary point , then we can introduce a local coordinate system f = ( x1 , . . . , xn ) : U ^ Rn around f ( p ) , where n = dimN ( ! ) . By continuity of f , we can also find an open neighborhood V of p , such that f ( V ) C U . Then for any w G Uk ( N ) , we can write

w\u = <= Wi1 . . . ikdx11 A ■ ■ ■ A dxik ,

1<ii <••• <ik

hence

( f *w ) \v = <= f * ( wii . . . ik ) f * ( dxii ) A ■ ■ ■ A f * ( dxik )

1<ii <••• <ik

= <=  ( wii . . . ik o f ) d ( xi1 o f ) A ■■■A d ( xik o f ) .

1 <i i <••• <ik <n

and

( f * ( dw ) ) \v = <= d ( wii . . . ik of ) Ad ( xii of ) A — Ad ( xik of ) = d ( f * ( w ) ) .

1 <= ii < - - - <ik<n

□

Proposition Lie derivation and exterior derivation of differential forms commute , that is

Lx o d = d o Lx .

Proof . Using the usual notation , for any differential form w , we have Lx ( dw ) = d ( $* ( dw ) ) \t=o = ddtd ( T* ( w ) ) \t=o .

In the last expression , we can change the order of differentiation with respect to the parameter t and the exterior differentiation by Young's theorem .

Indeed if we write the one - parameter family of forms at = using local coordinates as

at ( p ) = e w»i . . . ifc ( p , t ) dx11 A - - - A dxik ,

1<ii      <n

then

e e d t3 ( ^ail . . . ik ( p , t ) dxj A dx11 A ■ ■ ■ A dxik

1<ii < - - - <ik <n j=1 n

= e e djj dtail . . . ik ( p , t ) dxj A dx11 A ■ ■ ■ A dxlk

1<ii < - - - <ik <nj=1

=d ( drt"t

Consequently , <= x ( da ) = j - d ( Tt* ( a ) ) |t=o = d ( ftTt* ( a ) |t=o ) = d ( <= xw ) . □

Proposition ( Cartan's Formula ) . Exterior differentiation and inte - rior product with a vector field are related to Lie derivation of differential forms by the formula

<= x = ix o d + d o ix .

Proof . The formula is true for functions f G QP ( M ) , since

<= x ( f ) = X ( f ) = df ( X ) = ix ( df ) = ix ( df ) + d ( ix f ) .

In the last step we used that ix is 0 on smooth functions .

If the formula holds for a form a , then it holds for da . Indeed ,

<= x ( da ) = d ( <= xa ) = d o ix o d ( a ) + d o d o ix ( a ) = d o ix o d ( a ) ,

on the other hand ,

( ix o d + d o ix ) ( da ) = ix o d o d ( a ) + d o ix o d ( a ) = d o ix o d ( a ) .

If the identity holds for the forms a G Llk ( M ) and n G Hl ( M ) , then it holds for their wedge product as well . Using the Leibniz rule for <= x , we get

<= x ( a A n ) = <= x ( a ) A n + a A <= x ( n )

= ix ( da ) A n + d ( ixa ) A n + a A ix ( dp ) + a A d ( ixn ) .

Computing the other side ( ix ◦ d + d o iX ) ( w A n )

= ix ( dw A n + ( —1 ) kw A drj ) + d ( ixw A n + ( — 1 ) kw A ixn )

= ix ( dw ) A n + ( — 1 ) + dw A ixn + ( — 1 ) ixw A dn + ( — 1 ) w A ix ( dn ) + + d ( ixw ) A n + ( —1 ) k - 1ixw A dn + ( —1 ) kdw A ixn + ( —1 ) 2kw A d ( ixn ) = ix ( dw ) A n + d ( ix w ) A n + w A ix ( dn ) + w A d ( ix n ) .

Since locally , in the domain of a chart , any differential form can be written as the sum of wedge products of functions and differentials of functions , the proposition is true .

The formula below can also be used as a coordinate free definition of the exterior differential of a differential form .

Proposition . If w G Uk ( M ) , and Xo , . . . , Xk are smooth vector fields on M , then

k

dw ( Xo , . . . , Xk ) = ^ ( —irXi ( w ( Xo , . . . , Xi , . . . , Xk ) ) +

i=0

+ ^ ( —1 ) i+j w ( [Xi , Xj ] , Xo , . . . , Xi , . . . , Xj , . . . , Xk ) .

$0<i<j<k$

Proof . We prove the formula by induction on k . For a smooth function f G QP ( M ) , the second sum is empty , and df ( X ) = X ( f ) , so the formula holds . Assume that the formula is true for ( k — 1 ) - forms . Using Cartan's formula

dw ( Xo , . . . , Xk ) = ( ixo dw ) ( Xi . . . , Xk )

= ( Lxow ) ( Xi . . . , Xk ) — d ( ixow ) ( Xi . . . , Xk ) .

( Lxow ) ( X1 . . . , Xk ) can be expressed by a Leibniz - type rule as follows

k

( Lxow ) ( Xi . . . , Xk ) = Lxo ( w ( Xi . . . , Xk ) ) — Y , w ( Xi , . . . , LxoX< , . . . , Xk )

i=1

k

= Xo ( w ( Xi . . . , Xk ) ) + y ( —1 ) kw ( [Xo , Xi] , Xi , . . . , Xi , . . . , Xk ) .

i=i

The induction hypothesis applied to the ( k — 1 ) - form iXo w provides

k

—d ( ix o w ) ( Xi . . . , Xk ) = ^ ( —i ) iXi ( ixo w ( Xi , Xk ) ) +

i=1

+ ^ ( — 1 ) i+j+1iXo w ( [Xi , Xj ] , Xi , . . . , X ( i , . . . , XCj , . . . , Xk )

1<i<j<k

k

= ^ ( —1 ) iXi ( w ( Xo , . . . , Xi , . . . , Xk ) ) +

i=1

+ ^ ( —1 ) i+j w ( [Xi , Xj ] , Xo , . . . , Xi , . . . , Xj , . . . , Xk ) .

1<i<j<k

Adding the last two equations we obtain the statement for w .

The angle between osculating k - planes

Let y : I ^ Rn be a curve of general type in Rn parameterized by arc length . Fix a point to € I and for to + s G I , denote by ak ( s ) the angle between the osculating k - planes of y at to and at to + s for every

k < n .

Proposition . As the parameter to + s G I tends to to , the quotient ak ( s ) / |s| tends to |«fc ( to ) | .

Proof . The direction space of the osculating k - plane at t G I is spanned by the first k Frenet vectors ti ( t ) , . . . , tk ( t ) . Thus , the angle ak ( s ) is the angle between the unit k - vectors ( t1 A • • • A tk ) ( t0 ) and ( t1 A • • • A tk ) ( t0 + s ) . By elementary geometry , if an isosceles triangle has two equal sides of unit length enclosing an angle a , then the length of the third side is 2sin ( a / 2 ) . Applying this theorem gives

( ti A • • • A tk ) ( to + s ) — ( ti A • • • A tk ) ( to )

|s| sin ( afc ( s ) / 2 )

When s tends to 0 , the angle ak ( s ) also tends to 0 , so

ak ( s ) / 2

lim

s^o sin ( ak ( s ) / 2 ) As a corollary , we get

lim Oiisl = lim

s^o |s| s^o= || ( ti A •••A tk ) ' ( to )

Computing the derivative with the Leibniz rule and applying the Frenet formulae

( ti A • • • A tk ) ' = ti A • • • A ti A • • • A tk

i=i

k

= ( Kit2 A t2 A ••• A tk + ^ ti A — A ( Kiti+i — Ki - iti - i ) A ••• A tk .

=2

As the wedge product of vectors vanishes if two of the vectors are equal , the only nonzero summand on the right - hand side occurs at i = k

( ti A • • • A tk ) ' = ti A • • • A tk - i A ( Kktk+i )

Combining all we ave

lim = || ( ti A •••A tfc ) ' ( to ) y

s^Q |sK ( to ) | • II ( ti A • • • A tfc_i A tfc+i ) ( to ) | = |«fc ( to ) | , as it was to be proven .

Volume and Dihedral Angles of Inscribed Simplices

Let y : I ^ Rn be a curve of general type in Rn , k < n . Fix a point t G I and choose k +1 pairwise distinct parameters to , . . . , tk G I . Then the curve points y ( to ) , . . . , Y ( tfc ) span a k - dimensional simplex , the k - dimensional volume of which we denote by Vk ( to , . . . , tk ) . It is clear that if all the parameters to , . . . , tk tend to t simultaneously , then this volume tends to 0 .

Proposition . The limit of the volume Vk ( to , . . . , tk ) divided by the product of the edge lengths of the simplex spanned by the points Y ( to ) , . . . , Y ( tk ) as all the parameters ti tend to t , is equal to

( t ) • • • Kk - 1 ( t ) | .

Proof . The volume of a simplex can be obtained from the wedge product of the edge vectors running out from one of the vertices

Vk ( to , . . . , tk ) = kj || ( Y ( ti ) - Y ( to ) ) A - - - A ( Y ( tk ) - Y ( to ) ) l| .

We know that the mth order divided difference of y has the following explicit

During the proof of Theorem we showed that right - hand side is a 0 weight linear combination of the points y ( C ) . Consequently , equation also holds . Consider equation for m = 1 , . . . , k and take the wedge product of these k equations . Applying Proposition we get

Yl ( to , t1 ) A • • • A Yk ( ^ . . . , tk )

Taking the length of both sides and rearranging the equation gives

Vk ( to , . . . , tfc ) _ 1 ||7i ( *o , *i ) A ■ ■ ■ A yk ( to , . . . , tk ) ||

no<i<j<fc ii7 ( tj ) - y ( ti ) | no<i<j<k iiYi ( ti , tj ) | .

As we know the limits of the higher order divided differences as all the tj's tend to t , we can compute the limit of the right - hand side easily

Vk ( to , . . . , tk ) _! ( A !_ \ | ( Y; A ■ ■ ■ A Y ( k ) ) ( t ) ||

no<i<j<k IIY ( tj ) - Y ( ti ) | k! U=1 m7 ||Y / ( t ) | ( k+1 ) k / 2 .

|Y / A ■ ■ ■ A Y ( k ) || _ |Ak| _ K ••• «k| _ v ( k+1 ) k / 2|Kk - 1 . Kk - 2 . . . Kk - i| ,

which completes the proof .

By elementary geometry , if the area of a triangle is A and its side length are a , b , c , then the radius R of the circumcircle of the triangle is expressed by the formula

abc

_ 4A .

Thus , our theorem for k _ 2 has the following corollary .

Corollary . For a curve Y: I ^ Rn of general type in Rn , and for any three distinct points to , tp t2 in I , denote by R ( to , t1 , t2 ) € [0 , to] the circumradius of the triangle Y ( to ) Y ( t1 ) Y ( t2 ) . Then

lim R ( to , t1 , t2 ) _ |—Ay .

to , ti , t2^t |«1 ( l ) |

Though for k > 2 the limit in Proposition involves several curvatures , it is possible to construct expressions of the volumes of faces of various dimensions of a simplex whose limit gives one single curvature . An example of such a formula is Darboux's formula for the second curvature .

Corollary ( Darboux ) . Using the notations of Proposition

lim H V3 ( to , t1 , t2 , t3 ) _ |K2 ( t ) | .

to , ti , t2 , t3^t 2 yj Vr2 ( t1 , t2 , t3 ) Vr2 ( to , t2 , t3 ) V2 ( to , t1 , t3 ) V2 ( to , t1 , t2 )

Consider again a curve y : I ^ Rn of general type in Rn ,

and let 2 < k < n be an integer . Fix a point t , € I and if k _ n ,

then suppose that Kn - 1 ( t ) _ 0 . By Proposition when the points to , . . . , tk € I are sufficiently close to t , the simplex S spanned by y ( to ) , . . . , Y ( tk ) will be non - degenerate . Consider two facets of S , say the facet F spanned by the points y ( fo ) , . . . , y ( tfc - i ) and F spanned by y ( t1 ) , . . . , Y ( tk ) . These two facets intersect along a ( k — 2 ) - dimensional face L = F n F1 . Let a = a ( to , . . . , tk ) be the angle of the ( k — 1 ) - planes spanned by the facets F and F1 .

**Notes**

We remark that if 0 is the angle between the outer unit normals of the facets F and F1 , then the dihedral angle of the simplex at the ( k — 2 ) - dimensional face L is defined to be the complementary angle n — 0 . The angle a of the ( k — 1 ) - planes is the smaller of 0 and the dihedral angle 0 — 0 , that is , a = min{0 , n — 0} G [0 , n / 2] .

Proposition / / the distinct parameters to , . . . , tk G I tend to t simul - taneously , then

y a ( to , . . . , tk ) = Kk - i ( F )

Vi:™ f y7 ( tk ) — Y ( to ) || = ~ .

Proof . Denote by P the vertex y ( tk ) , by P' is orthogonal projection onto the ( k — 1 ) - plane containing F , and by P" its orthogonal projection onto the ( k — 2 ) - plane spanned by L . The triangle PP'P"A is a right triangle with right angle at P' and angle a at P" . For this reason , sin ( a ) = - ppy . PP' is the height of the original simplex corresponding to the facet F , PP" is the height of the simplex F corresponding to its face L . Thus , we can compute the lengths of these segments from the volume formula for simplices:

PP' = k • Vk ( to , . . . , tk ) , and PP" = ( k — 1 ) ■ Vk - i ( tl , . . . , tk ) .
Vk - l ( to , . . . , tk - l ) Vk - 2 ( ti , . . . , tk - l )

For simplicity , denote by , Ep , Ep , and the product of the edge lengths of the simplices S , F , F1 , and L respectively . It is clear that ES ■ EL = Ep ■ E_p ■ ||Y ( tk ) — Y ( to ) | . Consequently

sin ( a ) PP' ■ Ep ■ Ep

lY ( tk ) — Y ( to ) || = PP" ■ Es ■ El

Vk ( to , . . . , tk ) Vk - 2 ( tl , . . . , tk - l )

= k ■ es el

k — 1 ^ - l^ . . . , tk - l ) Vk - l ( tl , . . . , tk ) Ep E_p

Computing the limit of the right - hand side applying Proposition

sin ( a ( to , . . . , tk ) )

y7 ( tfc ) - y ( to ) y

1!2! ••• ( k - 1 ) ! • ( k - 1 ) !

The above equation shows that a tends to 0 as all the tj's tend to t , therefore a / sin ( a ) tends to 1 , so the equation implies the statement .

**Check your Progress 1**

Discuss Plane Curves

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Discuss Convex Curves

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## 11.6 LET US SUM UP

In this unit we have discussed the definition and example of Plane Curves , Convex Curves , Interior Product By A Vector Field , Exterior Differentiation

## 11.7 KEYWORDS

Plane Curves …. This section we deduce some facts on plane curves from the general theory of curves

Convex Curves…. A plane curve A simple closed plane curve 7 is convex

Interior Product By A Vector Field ….. The interior product of a smooth vector field X and a differential

Exterior Differentiation…. The differential df of a smooth function f G QP ( M ) is the differential 1 - form df G H1 ( M ) , the evaluation of which on a vector field X is given by df ( X ) = X ( f )

## 11.8 QUESTIONS FOR REVIEW

Explain Plane Curves

Convex Curves

## 11. 9 SUGGESTED READINGS

Differential Geometry, Differential Geometry & Application, Introduction to Defferential Geometry, Basic of Differential Geometry.

## 11.10 ANSWERS TO CHECK YOUR PROGRESS

Plane Curves

Convex Curves

( answer for Check your Progress - 1 Q )

# UNIT-12 : THE GAUSS—BONNET FORMULA

**STRUCTURE**



## 12.0 OBJECTIVES

After studying this unit , you should be able to:

- Understand about The Gauss—Bonnet Formula
- Steiner's Formula
- Minkowski's Formula

## 12.1 INTRODUCTION

Differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces Mathematical analysis of curves and surfaces had been developed to answer some of unanswered questions that appeared in calculus like the reasons for relationships between The Gauss—Bonnet Formula , Steiner's Formula , Minkowski's Formula

## 12.2 THE GAUSS—BONNET FORMULA

Consider a surface M C R3 in R3 . Define a local orthonormal frame e1 , e2 , e3 over an open subset U of M . Define the one forms a% and wj in the usual way . Then the Levi - Civita connection V of M is determined by the form w2 = —w2 through the relations

Vx e1 = wf ( X ) e2 , Vx e2 = w^ ( X ) e1 .

The structure equations for dw2 yield the equation

dw 2 = wj| A w 2 + w 2 A wf + w3 A wf = —w3 A w3 .

The form w3 A w3 is a multiple of the area form a1 A a2 of the surface M . The multiplier f in the equation

w3 A w3 = fa 1 A a2

can be obtained by evaluating both sides on the orthonormal system e1 , e2 . This gives

f = w3 A w| ( e 1 , e2 ) = det

According to equation the matrix on the right - hand side is the matrix of the second fundamental form of M with respect to the orthonormal basis ei , e2 , consequently , its determinant is the Gaussian curvature K of M . We conclude that

dw^ = A = Ka1 A a2 .

Applying Stokes' theorem this formula implies that fixing an orientation of U , and a regular domain D C U ,

Variants of this formula together with the geometrical interpretations of the second integral are called local Gauss - Bonnet formulae .

## 12.3 STEINER'S FORMULA

Steiner's formula , named after the Swiss geometer Jacob Steiner ( 1796 - 1863 ) , asserts that for a compact convex subset K of Rn , the volume of the parallel body

B ( K , r ) = {q € Rn | d ( q , K ) < r}

is a polynomial

of degree n of r . The constant Wj ( K ) appearing in the coefficient of rj is a geometrical invariant of the convex body K , called its jth quermassintegral . The number Vj ( K ) given by the equation wn - jVj ( K ) = ( — ) Wn - j ( K ) , where wn - j denotes the volume of the ( n — j ) - dimensional Euclidean unit ball , is the j th intrinsic volume of K .

For example , the constant term W0 ( K ) = Vn ( K ) is the volume of K . The coefficient 2Vn - 1 ( K ) = nW1 ( K ) of r is the surface volume of the boundary of K . The coefficient of rn - i is the average width of K multiplied with a constant depending only on the dimension . The coefficient wnV0 ( K ) = Wn ( K ) of rn does not depend on K , it is the volume of the n - dimensional unit ball for all K .

In this part we compute a smooth analogue of Steiner's formula . A practical formulation of the problem we want to solve is "How much paint we need to cover one side of a surface with a coat of paint of thickness r?" . As the layer of paint has positive thickness , it fills a part of space between the original surface M and a parallel surface of M lying at distance r from M .

Definition . Let M be a hypersurface in Rn , with unit normal vector field N and moving orthonormal frame e1 , . . . , en = N . For a real number r € R , we define the parallel surface Mr of M lying at distance r as the image of the map xr : M ^ Mr , xr = x + ren .

The following proposition summarizes some basic facts on the geometry of the parallel surfaces .

Proposition . The differential of the map xr : M ^ Mr has maxi - mal rank ( n — 1 ) at p € M if and only if 1 / r is not a principal curvature of M at p . If p is such a point , then the tangent spaces TpM and TXr ( p ) Mr are parallel;

Notes

the principal directions of M at p are also principal directions of Mr at xr ( p ) . If the principal curvatures of M and Mr are computed with respect to the unit normal vector field en , then if the principal curvature of M corresponding to the principal direction v € TpM is Ki , then the principal curvature of Mr corresponding to the same principal direction v € Txr ( p ) Mr is Ki / ( 1 — rKi ) .

Proof . The differential of the map xr is dxr = dx+rden = ^n - ei ( a1 +r^ln ) . In a coordinate free way , dxr takes the tangent vector v € TpM to the vector v — rLp ( v ) , where Lp is the Weingarten map of M at p € M . This means that the map dxr has maximal rank at p € M if and only if 1 / r is not a principal curvature of M at p . Assuming that xr has maximal rank , the tangent space of Mr at xr ( p ) is parallel to im ( ITpM — rLp ) = TpM . Thus , we can choose e1 o x - 1 , . . . , en o x - 1 as an orthonormal frame on Mr . By the chain rule , the Weingarten map of Mr at xr ( p ) is —d ( en o x - 1 ) ( xr ( p ) ) = Lp o ( IT M — rLp ) - 1 . This proves that if v is a principal direction of M with principal curvature Ki , then v is also a principal direction of Mr with principal curvature Ki / ( 1 — rKi ) .

Proposition . If D C M is a compact connected regular domain in M such that 1 / r is not a principal curvature of M at any point of D , then the volume measure of Dr = xr ( D ) C Mr can be expressed by the formula

( —1 ) i

Ki ( p ) dp )

R as the absolute value of a degree n — 1 polynomial of r , where Ki ( p ) denotes the ith elementary symmetric polynomial

Ko ( p ) = 1 ,

K1 ( p ) = K1 ( p ) + + Kn - 1 ( p ) ,

Kn - l ( p ) = Kl ( P ) • K2 ( p ) • • • Kn - l ( p ) of the principal curvatures Ki ( p ) , . . . , Kn - i ( p ) at p .

Proof . As the derivative of xr at p is the linear map ( IT M — rLp ) , the pull - back of the volume form of Mr to M is det ( / Tx ( ) M — rLx ( . ) ) a1 A • • • A a" - 1 . For this reason ,

Pn - i ( Dr ) = / | det ( / TpM — rLp ) |da ( p ) ,

J D

where the integral is taken with respect to the volume measure a of M . The integrand can be expressed with the help of the characteristic polynomial of Lp as

det ( / TpM — rLp ) ) = rn 1 • det ^1 Itpm — Lp^j = rn 1 ]—[ ^1 — K□ ( p ) ^

n—1 n—1

= rn—1<= ( —1 ) % ( p ) zn—i—i = e ( "i ) 'K' ( p ) r' .

=0 i=0

Since 1 / r is not a principal curvature at any point of D , det ( ITpM — rLp ) does not vanish on D , therefore , as D is connected , it has constant sign . As a corollary we obtain that the integral of its absolute value is the absolute value of its integral . Combining these observations ,

J det ( lTpM — rLp ) da ( p ) = ^ ( — 1 ) □ ^ K□ ( p ) dp^ A

as we wanted to prove .

The layer of thickness r over the regular domain D C M , lying between D and Dr is parameterized by the map

h: M x [0 , r] ^ Rn , h ( p , s ) = p + sen ( p ) .

We claim that if D is compact and r > 0 is sufficiently small , then h is a diffeomorphism . Using the natural decomposition T ( p , s ) ( M x [0 , r] ) = TpM □ Ts[0 , r] , the tangent space T ( p , s ) ( M x [0 , r] ) is spanned by the vectors e1 ( p ) , . . . , en—1 ( p ) and d ( s ) , where the vector field d on R is the unit vector field corresponding to the derivation of functions with respect to their single variable .

The images of these tangent vectors under the derivative map of h are ei ( p ) — sLp ( ei ( p ) ) , . . . , en - 1 ( p ) — sLp ( en - 1 ( p ) ) and en ( p ) . These vectors are linearly independent if 1 / s is not a principal curvature of M at p . If M has a positive principal curvature at a point p of D , then let

k+ = max max kAp )

pED 1<i<n - 1

be the maximal value of principal curvatures at all points of D , otherwise set k+ = 0 . From the previous arguments , if 0 < r < 1 / k+ , then the derivative of h has maximal rank n at each point of D x [0 , r] , hence h is a local diffeomorphism by the inverse function theorem . Suppose that h is not a diffeomorphism for any r > 0 . Then , since h is a local diffeomorphism for small values of r , the reason why it is not a diffeomorphism for these r's , is that h is not injective . If h is not injective for the r = 1 / 2 , 1 / 3 , . . . , 1 / k , . . . , then we can find two sequences of pairs ( pk , sk ) = ( p*k , s*k ) such that pk , p*k G M , sk , sk G ( 0 , 1 / k ) , and pk + sken ( pk ) = p*k + s*ken ( p*k ) . Since D is compact , there is a convergent subsequence of the sequence pk , say pki which tends to p as i tends to infinity . Then the pairs ( pki , ski ) and ( pk , s*ki ) tend to the pair ( p , 0 ) But then ( p , 0 ) would not have a neighborhood on which h is injective . This contradicts that h is a local diffeomorphism .

Proposition . Using the above notation , if h is a diffeomorphism the volume of the layer of thickness r over D parameterized by h is equal to the polynomial

Proof . Introduce on M x [0 , r] the Riemannian metric , in which the basis e1 ( p ) , . . . , en - 1 ( p ) , d ( s ) is orthonormal . The pull - back of the volume form of Rn by h is the form

± det ( / TpM — sLp ) ) ^1 A ••• A dan 1 A ds ,

where the sign depends on the orientations chosen . Thus , and the absolute value can be omitted . We conclude

fn— 1

•M u = / {n - »' ( JD KMd'> ) d = e m— ( X Kk - 1 MdOri ) rk ,

as it was to be proven . □

Exercise . Prove that the above proposition implies Steiner's for - mula for the volume of parallel bodies of a convex body K , the boundary M = dK of which is a smooth hypersurface in Rn . Show that we get the following explicit expression for the coefficients of the polynomial:

An ( B ( K , r ) ) = An ( K ) + l ( / 1 Kk - 1 ( p ) |d^ rk . q

b—1 \J D /

Exercise . Show that if K is a convex body in Rn bounded by a smooth hypersurface M with Gauss - Kronecker curvature function

K = Kn - i , then

|K ( p ) \da ( p ) = n^n ,

' M

where wn is the volume of the n - dimensional unit ball ( consequently , nwn is the surface volume of the unit sphere Sn - 1 in Rn ) .

## 12.4 MINKOWSKI'S FORMULA

Being important geometrical invariants of a convex body , the coefficients of the polynomial in Steiner's formula are studied thoroughly . In this part , we focus on the coefficient rn - 1 , which is , up to some constant multiplier , the integral of the ( n — l ) st elementary symmetric polynomial of the principal curvatures of the boundary hypersurface of K . Hermann Minkowski ( 1864 - 1909 ) proved that if K is a compact convex set in Rn , then this coefficient is the average width of K up to some constant , depending only on the dimension n .

We prove below a formula which is true for any compact hypersurface M in Rn which , in the special case when M is the boundary of a convex set reduces to Minkowski's formula .

Proposition ( Minkowki's Formula ) . Let M be a smooth compact hypersurface in Rn , en: M ^ Rn be a unit normal vector field on M . Let pn: M ^ R , pn ( p ) = ( en , p ) denote the signed distance of the origin from

the affine tangent space of M at p . Then denoting by a the surface volume measure on M , we have

^ / Kn - 2da = f PnKn - ida ,

n - 1 J M JM

where the elementary symmetric polynomials Ki of the principal curvatures of M taken with respect to the unit normal vector field en are defined .

Proof . Let x: M ^ Rn be the inclusion map , and choose a positively oriented orthonormal frame ei , . . . , en over an open subset U of M and orient M so that the vector fields ( ei , . . . , en - i ) give positively oriented bases at points of U . Consider the functions pi: U ^ R , pi ( p ) = ( ei , p ) . Using the fundamental equations of hypersurfaces , we can express the differentials of the functions pi by the equations

dpi = ( dei , x ) + ( ei , dx )

= fa=1 Pj —I + ai

\ eju Pj —n ,

Define the differential ( n — 2 ) - form n on U as

n—1

n = ^ ( —! ) i+1Pi - n A ••• A - n A ••• A - n—1 ,

i=1

where the hat above — n means that — n is omitted . Compute the differential of n . Clearly , n—1

dn = ^ ( - l ) i+1dpi A —n A • • • A —n A • • • A —1_ 1 +

i=1 n—1

+ ^ ^ ( - l ) i+jpi—n A • • • A d—n A • • • A —n A • • • A —rf—1 +

i= 1 1 <j<i n— 1

+ ^ ^ ( - 1 ) i+j —1pi—n A •••A —n A •••A d—n A •••A —n—1 ,

i=1i<j<n—1

hence

n—1 n

= e ^ ( - 1 ) i+1pjwj A w1 A • • • A w? A • • • A w?—1 +

i=1 j=1 n— 1

+ E ( - 1 ) i+v a w? a ••• a w? a ••• a ^n—1+

i=1

n— 1 n

+ e e e ( - ! ) i+jpiwna - - - a ( Wkaw? ) a^aw? a^aw?—1+

i=1 1<j<ik = 1 n— 1 n

+ E E E ( - ! ) i+j—1Pi^ n A •••A ^ A •••A ( w ) A w? ) A •••A w1—1 .

i=1i<j<n—1k=1

The sum of the last two terms can be simplified as

n—1

e e ( - ! ) i+jPiwn A • • • A ( wj A w? ) A • • • A W? A^A w?—1 +

i=1 1<j<i n—1

+ e e ( - 1 ) i+j—1PiWn A •••A W? A •••A ( wj A w? ) A •••A w?—1

i=1i<j<n—1

n— 1

= e e ( - 1 ) ( i+j ) + ( j—1 ) + ( i—j—1 ) piwj A w? A ••• A w? A ••• A w?—1 +

i=11<j<i n— 1

+ e e ( - 1 ) ( i+j —1 ) + ( j—2 ) + ( j—i—1 ) piwj A w? A •••A W? A •••A w?—1

i=1i<j<n—1 n— 1 n— 1

= e e ( - 1 ) jPiwj A w? A •••A w? A • • • A w?—1 .

i=1 j = 1

Substituting back into the last formula for dn we obtain

n— 1

dn =Pn ^^ ( —1 ) i+1wr A wi" A • • • A w" A • • • A w1—1+

i=1

n—1

+ E ( - 1 ) i+1ffi A w? A ••• A w? A ••• A w?—1

i=1

n— 1

= ( n - 1 ) p?wn A ••• A w?—1 + e ( - 1 ) i+1^i A wn A ••• A w? A ••• A w?—1 .

i=1

The forms n , Pn • w" A • • • A w"—1 and J2"=— ( —A w" A • • • A w™ A • • • A w"_ i are natural forms on M in the sense that they do not depend on the choice of the orthonormal frame . Indeed , if v1 , . . . , vn—2 € TpM , then nP ( vi , . . . , vn—2 ) is the signed volume of the parallelepiped spanned by the orthogonal projection n ( p ) = 5^"= / Pi ( p ) ei ( p ) of p onto TpM and Lp ( v1 ) , . . . , Lp ( vn—2 ) , where Lp is the Weingarten map at p . The differential formpnAw"A^ • •Aw"_ 1 is the volume form of M multiplied by the product of the Gauss - Kronecker curvature Kn - 1 and the signed distance pn of the origin from the tangent plane . Naturality of the third form follows from equation and naturality of the first two forms . Computing the third form using a frame

whose vectors point in principal directions at a point , we see easily that

n—1

^^ ( —1 ) i+1ai A w" A • • • A w™ A • • • A w"—1 = Kn—2a1 A • • • A an—1 .

i=1

The importance of naturality is that although the forms n , pn • w" A • • •A w"_ 1 and L^ - l^V A w" A • • • A w" A • • • A w"—1 were defined locally , on an open set , on which a local orthonormal frame e1 , . . . , en can be defined , these forms are globally defined on M and relation also holds globally on M . If M is a compact hypersurface ( with no boundary ) , then by Stokes' theorem

/ dn = n = 0 ,

Jm Jbm

thus , integrating we obtain the formula

^ [ Kn—2da = / p™Kn—1da

n - 1 Jm Jm

we wanted to show .

Exercise . Prove that for a compact hypersurface M C Rn ,

I enKn— 1d& = 0 . p .

Jm

Hint: Apply Minkowski's formula with different choices of the origin of the coordinate system .

Notes

In the special case , when M is strictly convex and the Gauss map en: M ^ Sn—1 is a diffeomorphism between M and Sn—1 , the second integral can be rewritten as an integral on the sphere .

Since the derivative map Tpen: TpM ^ Ten ( p ) Sn 1 is — Lp , the pull - back of the volume form of Sn - 1 by en is ( —l ) n - 1Kn - 1a1 A - • •A an - 1 . Consequently ,

Definition . The width wK ( u ) of a compact set K C Rn in the direction u G Sn - 1 is the width of the narrowest slab bounded by hyperplanes orthogonal to u that contains K . The width is given by the formula

wK ( u ) = max ( u , x ) — min ( u , x ) .

xGK xGK

Definition . The mean width w ( K ) of a compact set K C Rn is the average of its widths in different directions , that is

If K is a compact convex regular domain with boundary M = dK , and en: M ^ Rn is the exterior unit normal vector field along M , then for u G Sn - 1 , ( u , x ) attains its maximum ( or minimum respectively ) on K at the points x G M of the boundary , at which en ( x ) = u ( or en ( x ) = — u re - spectively ) . At such a point ( u , x ) = pn ( x ) ( or ( u , x ) = —pn ( x ) respectively ) . In the special case , when the Gauss map is a diffeomorphism between M and Sn - 1 , we get

Thus , we get the following corollary of Proposition also known as Minkowski's Formula .

Corollary ( Minkowski's Formula on the Mean Width ) . Let K C Rn be a compact convex regular domain with boundary M = dK , such that the exterior unit normal vector field en : M ^ Rn along M is a diffeomorphism between M and Sn - 1 . Then the coefficient of rn - 1 in the polynomial expressing the volume of B ( K , r ) is equal to the following quantities

Rigidity of Convex Surfaces

Cauchy's rigidity theorem for convex polytopes says that the shape of a 3 - dimensional convex polytope is uniquely determined by the shapes of the facets and the combinatorial structure describing which are the common edges of the neighboring facets . More formally , if we have two convex 3 - dimensional polytopes Pi and P2 and a bijection 4: 3P\ ^ dP2 , which maps each facet of Pi isometrically onto a facet of P2 , then 4 extends to an isometry of the whole space R3 .

We want to prove a smooth analogue of Cauchy's rigidity theorem here . In - stead of two convex polytopes , we shall consider two convex compact regular domains , Ki and K2 in R3 , and require that the bijection 4: dKi ^ dK2 between their boundaries be a bending . This means that 4 should preserve the lengths of curves lying on the boundary of Ki .

Let M be a smooth hypersurface in Rn , ei , . . . , en be a local orthonormal frame on an open subset U C M , h: M ^ M be a bending of M . Then the images of ei , . . . , en - i under the derivative of h yield an orthonormal tangential frame ei , . . . , en - i along h ( U ) , which can be extended uniquely to an orthonormal frame ei , . . . , en having the same orientation as ei , . . . , en . Using this frame on M , we can introduce the differential 1 - forms ed , and the functions p^ , in the usual way . Instead of working with these differential forms and functions , we prefer to work with their pull - back forms ed = h*aj , wj = h'wj and the functions pj = pj o h . Since h is an isometry of Riemannian manifolds , = cd . The Riemannian metric also determines the Levi - Civita connection , which is encoded by the one forms wj for 1 < i , j < n — 1 , so we also have wj = wjj for 1 < i , j < n — 1 .

As the following proposition claims , the exterior shapes of U and h ( U ) are determined by the forms w" and w" .

Proposition . If U is connected , then h\u extends to an orientation - preserving isometry of the whole space if and only if wj = w" for 1 < i < n—1 .

Proof . The easier part of the statement is that if the bending extends to an orientation preserving isometry of the space , then w" = w'n . We leave the details of this direction to the reader .

**Notes**

Assume now that wn = w" for all 1 < i < n — 1 . Choose a point p G U and an orientation preserving isometry 4 of Rn such that 4 ( p ) = h ( p ) and Tp4 ( ej ( p ) ) = ej ( h ( p ) ) for 1 < i < n — 1 . Then h = 4 - i o h is a bending of M which fixes p and the frame ei ( p ) , . . . , en - i ( p ) . Denote by ei , . . . , en the frame induced on h ( U ) in the same way as the frame ei , . . . , en was obtained on h ( U ) .

Let y: [0 , 1] ^ U be a smooth curve starting from y ( 0 ) = p . Consider the collection of the vector valued functions ( ei o y , . . . , en o y ) : [0 , 1] ^ Rn .

It satisfies the system of differential equations

d n

dt 0 Y ) ( t ) =Y1 wj ( Y' ( T ) ) ( el 0 Y ) ( t ) , ( 1 < i < n )

j=1

The same system of differential equations is satisfied also by the vector valued function ( ei o h o 7 , . . . , en 0 h 0 7 ) : [0 , 1] ^ Rn . Both solutions start from the same initial vectors at 0 , therefore , by the uniqueness of the solutions of ordinary differential equations with given initial value , ei o 7 = ei o h o 7 for i = 1 , . . . , n . From this we obtain

n —1 n —1

Y' ( t ) = ^ ( t ) ) ei ( Y ( t ) ) = ^ ^i ( 7 / ( T ) ) ei ( h ( Y ( t ) ) ) = ( h o Y ) / ( t ) ,

i=1 i=1

which implies by y ( 0 ) = h ( Y ( 0 ) ) = p that y = h o y . Since U is connected , y ( 1 ) can be any point in U , consequently h$—1 o h = idM , that is , T extends h|U to an isometry of the whole space . □

The main tool of proving the rigidity of convex surfaces in R3 is a gener - alization of Minkowski's formula due to Gustav Herglotz ( 1881 - 1953 ) . This formula has the same form as Minkowski's formula , but some of the "ingre - dients" of the formula are taken not from the hypersurface M but from a bending M of M . This way , Herglotz's formula relates the geometries of M and M to each other , and in the special case , when M = M , it returns Minkowski's formula .

Proposition ( Herglotz's Formula ) . Let M be a compact hypersurface in Rn , h: M ^ Rn be a bending , h ( M ) = M . Starting with a local orthonormal frame ep . . . , en over an open subset U of M define the forms wj , wj and the functions Kn—2 , pi and pi as above . Then the forms

pn ( sn - wn A • • • A w" A • • • A wn—1^ and Kn—2 • a1 A • • • A an— 1 do not de - pend on the choice of the frame , consequently they are properly defined over M , and A wn A •••A wn—1 ) = f Kn—2 • a1 A •••A an— 1 . J Jm

Proof . The proof follows the same line of computation as the proof of Min - kowski's formula , the only difference is that some terms are marked with a bar to indicate that they come from the bent surface MM . Due to the similarity of the two proofs , we skip some details . Define the differential ( n — 2 ) - form

n on M by the equation

n— 1

n = e ( - ! ) i+1^n A ••• A ^ A ••• A ^n—1 ,

i=1

and compute its differential as

n— 1

dn = ^ ( - l ) i+1dpi A w" A ••• A w" A ••• A w"—1 +

i=1 n— 1

+ ee ( - 1 ) i+jj9jwn A • • • A dw" A • • • A w" A • • • A w"— - L +

i=1 1<j<i

n— 1

+ e e ( - 1 ) i+j —1J4iwn A •••A Wn A •••A dw" A •••A w"—1 ,

**Notes**

i=1i<j<n—1

hence

n—1 n

dn = e e ( - 1 ) i+1PJj Wj A w" A - - - A W" A^A w"—1+

i=1 j = 1 n— 1

+ e ( - 1 ) i+1ffi A w" A • • • A w" A • • • A w"—1 +

i=1

n— 1 n

+ e e e ( - 1 ) i+jpw" a • • • a ( wk a wn ) a • • • a w" a • • • a w"—1+

i=1 1<j<i k=1 n— 1 n

+ e e e ( - 1 ) J+j —1piw" A - - - A wn A • • • A ( w| A w" ) A^A w"—1

i=1 i<j<n—1k=1

The sum of the last two terms can be simplified to

n—1 n—1

e e ( - 1 ) jpwj A w" A ••• A w" A ••• A w"—1 .

i=1 j=1

Substituting back we obtain dn = pn ^e w" A • • • A w" A • • • A w"—^ + Kn—2 • ff1 A • • • A an—1 .

The forms n , p>n i , 1 w"A - • •Aw"A^ • •Aw"— 1 and Kn—2a1 A - • •Aan—1 do not depend on the choice of the orthonormal frame e1 , . . . , en—1 , hence they are

defined globally . For example , if vi , . . . , vn - 2 & TpM , then nP ( v 1 , . . . , vn - 2 ) is the signed volume of the parallelepiped spanned by ( Tph ) - 1 ( n ( h ( p ) ) ) and Lp ( v1 ) , . . . , Lp ( vn - 2 ) , where n ( h ( p ) ) is the orthogonal projection of h ( p ) onto Th ( p ) M , Lp is the Weingarten map at p . Naturality of the third form was obtained during the proof of the Minkowski formula , that of the second form follows form the relation between the three forms .

If M is a compact hypersurface with no boundary , then integrating we obtain Herglotz's formula .

Theorem . Let M and MI be the boundary surfaces of the compact convex regular domains C and C in R3 . Assume that M has positive Gaussian curvature K2 > 0 and that there is a bending h: M ^ M . Then h extends to an isometry $ of the whole space . In particular , C and C are congruent .

Proof . Orient M and M so that augmenting a positively oriented basis of one of the tangent spaces by the exterior unit normal of C or C respectively give a positively oriented basis of the space R3 . We may assume without loss of generality the h: M ^ M is orientation preserving , otherwise consider instead of h a composition of h with a reflection in a plane .

Let us use the same notations as in Minkowski's and Herglotz's formulae . Translating C and C we may assume that the origin is in the interior of C and C . Then p3 > 0 and p3 > 0 .

The integral of the Minkowski curvature H = K1 / 2 of M over M can be expressed both by Minkowski's and Herglotz's formulae as

of the volume form a1 A a2 . We show that the function f is non - positive . Fix a point q G M and choose a positively oriented orthonormal basis v1 , v2

in Tq M then

' ^f ( vi ) wf ( v2 ) ' . wf ( vi ) wf ( v2 )

is the matrix of the second fundamental form and the Weingarten map of M at q with respect to the basis ( v1 , v2 ) . For this reason , B is negative definite , and has determinant K2 ( q ) . Similarly ,

wf ( vi ) wf ( v2 ) . wf ( v 1 ) wf ( v2 ) y

**Notes**

is the matrix of the second fundamental form and the Weingarten map of M at h ( q ) with respect to the basis ( Tqh ( v1 ) , Tqh ( v2 ) ) , and it is also negative definite , and has determinant K2 ( q ) .

Consider the quadratic polynomial

D ( c ) = det ( B - cB ) = K2 ( q ) ( c2 — tr ( BB - 1 ) c + 1 ) .

Since B is negative definite , B — cB is positive definite for large c , consequently , D must have a positive root c0 > 0 . Since the product of the two roots is 1 , the second root is 1 / c0 . Then D ( c ) = K2 ( q ) ( c — c0 ) ( c — 1 / c0 ) and

f ( q ) = D ( 1 ) = —K2 ( q ) ( 1 — c0 ) 2 < 0 .

co

Equality f ( q ) = 0 holds if and only if tr ( BB - 1 ) = 2 . But we also know that det ( BB - 1 ) = 1 , hence f ( q ) = 0 if and only if both eigenvalues of BB - 1 are equal to 1 . Since both B and B are symmetric , and B is negative definite , BB - 1 is diagonalizable because of the principal axis theorem . This way , the eigenvalues of BB - 1 are equal to 1 if and only if BB - 1 = I and B = B . As a corollary , we obtain

f Hda —f Hda = 1 / P3 ( ^f — ^f ) A ( w3 — ^2 ) = ^ / Pzfa1 A a2 < 0 ,
JM JM 2 J M 2 J M

which implies

/ Hda < Hda .

'M JM

However , the role of M and M is symmetric , so the reversed inequality

/ H da > / Hda ' M JM

must also be true , therefore we must have equality in both inequalities . Equal - ity implies f = 0 , which can hold only if the matrices B and B are equal for each q G M and each choice of viv2 G TqM , that is , if w3 = w3 and w3 = cf3 . However this condition is equivalent to the extendability of h to an orientation preserving isometry of the space R3 .

Geodesics

We define the length of a smooth curve 7: [a , b] ^ M lying on a Riemannian manifold ( M , ( , } ) to be the integral

It is worth mentioning that the classical definition of length as the limit of the lengths of inscribed broken lines does not make sense , since the distance of points is not directly defined . The situation is just the opposite . We can define first the length of curves as a primary concept and derive from it a so called intrinsic metric d ( p , q ) , at least for connected Riemannian manifolds as the infimum of the lengths of all curves joining p to q . The metric enables us to define the length of "broken lines" given just by a sequence of vertices P**i ,** . . . , $P_N$ to be the sum of the distances between consecutive vertices . There is a theorem saying that the length of a smooth curve 7: [a , b\ ^ M is equal to the limit of the lengths of inscribed broken lines 7 ( $t_0$ ) , Y ( ti ) , . . . , Y ( $t_N$ ) , a = $t_0$ < $t_1$ < ■ ■ ■ < $t_N$ = b as the maximum of the distances |tj — $t_{i-1}$| tends to zero .

To find the analog of straight lines in the intrinsic geometry of a Riemannian manifold we have to characterize straight lines in a way that makes sense for Riemannian manifolds as well . Since the length of curves is one of the most fundamental concepts of Riemannian geometry , we can take the following characterization: a curve is a straight line if and only if for any two points on the curve , the segment of the curve bounded by the points is the shortest among curves joining the two points . A slight modification of this property could be used to distinguish a class of curves , but it is not clear at first glance whether such curves exist at all on a general Riemannian manifold .

For a physicist a straight line is the trajectory of a particle with zero ac - celeration or that of a light beam . This observation can also give rise to a definition . We only have to find a proper generalization of "acceleration" for curves lying in a Riemannian manifold . It seems quite natural to proceed as follows . The speed vectors of a curve yield a vector field along the curve . On the other hand , by the fundamental theorem of Riemannian geometry , the Riemannian metric determines a unique affine connection on the manifold which is symmetric and compatible with the metric . In particular , one can differentiate the speed vector field with respect to the curve parameter and may call the result Y" = Vy Y' the acceleration vector field along the curve .

**Definition .** Let M be a Riemannian manifold , Y be a curve on it . We say that Y is a geodesic if

Vy Y' = 0 .

**Remark .** More generally , if ( M , V ) is a manifold with an affine connection , then the curves satisfying Vy Y' = 0 are said to be autoparallel . Geodesics are autoparallel curves for the Levi - Civita connection .

**Proposition .** The length of the speed vector of a geodesic is constant .

Proof . By the compatibility of the connection with the metric , parallel trans - port preserves length and angles between vectors . The definition of geodesics implies that the speed vector field is parallel along the curve , consequently consists of vectors of the same length .

The proposition follows also from the equality

<Y' , Y' ) ' = <Vy y' , y'> + <y' , Vy y'> =0 .

As a consequence , we get that the property of "being geodesic" is not invariant under reparameterization . The parameter t of a regular geodesic is always related to the natural parameter s through an affine linear transformation i . e . t = as + b for some a , b G R . This motivates the following definition .

**Definition .** A regular curve on a Riemannian manifold is a pre - geodesic if its natural reparameterization is geodesic .

In terms of a local coordinate system with coordinates ( $x^1$ , . . . , $x^n$ ) a curve 7 in the domain of the chart determines ( and is determined by ) n smooth functions 7 ==> = $x^1$ o 7 ( 1 < i < n ) . The equation Vy Y' = 0 then takes the form

n

$Y^k$ + r j o Y • Y ==> ' • $Y^{j'}$ = 0 for all 1 < k < n .

==> , j= 1

The existence of geodesics depends , therefore , on the solutions of a certain system of second order differential equations .

Introducing the new functions v ==> = 7 ==> this system of n second order differ - ential equations becomes a system of 2n first order equations

$$Y^{k\prime} = v^k ,$$

$v^k$ = rj o Y • v ==> • $v^j$ ,

==> , j=i

Applying the existence and uniqueness theorem for ordinary differential equations one obtains the following .

**Proposition .** For any point p on a Riemannian manifold M and for any tangent vector X G $T_pM$ , there exists a unique maximal geodesic 7 defined on an interval containing 0 such that 7 ( 0 ) = p and y' ( 0 ) = X .

If the maximal geodesic through a point p with initial velocity X is defined on an interval containing [—e , e] then there is a neighborhood U of X in the tangent bundle such that every maximal geodesic started from a point q with initial velocity Y G $T_qM$ is defined on [—e , e] .

Since a geodesic with zero initial speed can be defined on the whole real straight line , for each point p on the manifold one can find a positive S such that for every tangent vector X G $T_pM$ with ||X|| < S , the geodesic defined by the conditions 7 ( 0 ) = p , 7' ( 0 ) = X can be extended to the interval [0 , 1] . The following notation will be convenient . Let X G $T_pM$ be a tangent vector and suppose that there exists a geodesic 7: [0 , 1] ^ M satisfying the con - ditions 7 ( 0 ) = p , Y' ( 0 ) = X . Then the point 7 ( 1 ) G M will be denoted by $\exp_p$ ( X ) and called the exponential of the tangent vector X .

Using the fact that for any c G R , the curve t ^ 7 ( ct ) is also a geodesic we see that the geodesic 7 is described by the formula

$$Y(t) = \exp_p(tX) .$$

As we have observed , $\exp_p$ ( X ) is defined provided that ||X|| is small enough . In general however , $\exp_p$ ( X ) is not defined for large vectors X . This motivates the following .

**Definition .** A Riemannian manifold is geodesically complete if for all p G M , $\exp_p$ ( X ) is defined for all vectors X G $T_pM$ .

This is clearly equivalent to the requirement that maximal geodesics are defined on the whole real line R .

**Proposition .** For a fixed point p G M , the exponential map $\exp_p$ is a smooth map from an open neighborhood of 0 G $T_pM$ into the manifold . Furthermore , the restriction of it onto a ( possibly even smaller ) open neighborhood of 0 G $T_pM$ is a diffeomorphism .

Proof . Differentiability of the exponential mapping follows from the theorem on the differentiable dependence on the initial point for solutions of a system of ordinary differential equations . To show that $\exp_p$ is a local diffeomor - phism , we only have to show that its derivative at the point 0 G $T_pM$ is a non - singular linear mapping ( see Inverse Function Theorem ) . Since $T_pM$ is a linear space , its tangent space $T_0$ ( $T_pM$ ) at 0 can be identified with the vector space $T_pM$ itself . Through this identification , the derivative of the exponential map at 0 maps $T_pM = T_0$ ( $T_pM$ ) into $T_pM$ . We show that this derivative is just the identity map of $T_pM$ , hence non - singular .

Let X be an element of the tangent space $T_pM = T_0$ ( $T_pM$ ) . To determine where X is taken by the derivative of the exponential mapping , we represent X as the speed vector of the curve t ^ y ( t ) = tX at t = 0 . The exponential mapping takes this curve to the geodesic curve 7 = $\exp_p$ oy , 7 ( t ) = $\exp_p$ ( tX ) , the speed vector of which at t = 0 is X , so the derivative of the exponential map sends X to itself and this is what we claimed .

By the proposition , we can introduce a local coordinate system , based on geodesics , about each point of the manifold as follows . We fix an orthonormal basis in the tangent space $T_pM$ , which gives us an isomorphism i: $T_pM$ ^ $R^n$ that assigns to each tangent vector its components with respect to the basis , and then take i o $\exp^{-1}$ . The map i o $\exp^{-1}$ is a diffeomorphism between an open neighborhood of p and that of the origin in $R^n$ , therefore , it is a smooth chart on M . Coordinate systems obtained this way are called normal coordinate systems , while we shall call the inverse of them normal parameterizations .

For a Riemannian manifold M , we can define the sphere of radius r centered at p G M as the set of points q G M such that d ( p , q ) = r , where d ( p , q ) denotes the intrinsic distance of p and q . When the radius of the sphere is increasing , the topological type of the sphere changes at certain critical values of the radius . For small radii however , the intrinsic spheres are diffeomorphic to the ordinary spheres in $R^n$ , and what is more , we have the following .

**Theorem .** The normal parameterization of a manifold about a point p maps the sphere about the origin with radius r , provided that it is contained in the domain of the parameterization , diffeomorphically onto the intrinsic sphere centered at p with radius r .

Formula for the First Variation of the Length

**Definition .** A variation of a smooth curve 7: [a , b] ^ M is a smooth mapping 7* from the rectangular domain [—S , J] x [a , b] into M such that Y* ( 0 , t ) = Y ( t ) for all t G [a , b] .

Given a variation of a curve we may introduce a one parameter family of curves $Y_e$ , e G [—S , S] by setting $7_e$ ( t ) = 7* ( e , t ) . By our assumption , these curves yield a deformation of the curve 70 = 7 .

**Theorem** ( Gauss Lemma ) . Let M be a Riemannian manifold , p € M , and denote by $S_r$ the sphere of radius r in $T_pM$ centered at the zero tangent vector . Assume r is chosen to be so small that the exponential mapping is a diffeomorphism on a ball containing $S_r$ and denote the exponential image of $S_r$ by $S_r$ . Then for any X € $S_r$ the radial geodesic t ^ $\exp_p$ ( tX ) is perpendicular to $S_r$ .

Proof . Every tangent vector of $S_p$ can be obtained as the speed vector of a curve $\exp_p$ oft where ft is a curve in $S_r$ passing through , 0 ( 0 ) = X . Given such a curve , let us define a variation of the geodesic 7: t ^ $\exp_p$ ( tX ) in the following way

7 . ( e , t ) := expp ( tft ( e ) ) .

For a fixed e , the curve is a geodesic of length r so / ( E ) is constant . Thus , the previous theorem implies that

0 =f ( 0 ) = ( ai - " ( 0 , " , R11 ) - H" . 0' , ^ ) .

Since y . ( e . 0 ) = $exp_p$ ( 0 • ft ( e ) ) = p and y . ( e . 1 ) = $exp_p$ ( ft ( e ) ) , we have $d_1$Y . ( 0 . 0 ) = **0** and $5_1$y . ( 0 . 1 ) = ( $exp_p$ oft ) ' ( 0 ) , therefore , we get ° = ( ( ex pp oft ) , ( 0 ) .

showing that 7 intersects $S_r$ orthogonally .

Now we are ready to prove the theorem saying that $S_r$ is a sphere in the intrinsic geometry of the manifold . It is clear that d ( p , q ) < r for any point q on $S_r$ , since the radial geodesic from p to r has length r , so all we need is the following .

**Theorem .** If 7: [a , b] ^ M is an arbitrary curve connecting p to a point of $S_r$ , then its length is > r .

Proof . We may suppose without loss of generality that S ( b ) is the only in - tersection point of the curve with $S_r$ and S ( t ) = P for t > a . Then there is a unique curve 7 in the tangent space $T_pM$ such that 7 = $exp_p$ 07 . Let **N** denote the vector field on $T_pM$ \ **{0}** that is the gradient vector field of the function f: X i© - ||X|| on $T_pM$ , and therefore consists of unit vectors per - pendicular to the spheres centered at the origin . The theorem above shows that the derivative of the exponential map takes **N** into a unit vector field **N** on M , perpendicular to the sets $S_t$ .

We can estimate the length of a curve as follows

p b p b

l ( S ) = |S' ( r ) |d r > ( 7 ' ( t ) , N ( S ( T ) ) ) dT .

J a J a

Since ( S' ( T ) , I S ( S ( T ) ) is the component parallel to **N** ( q ) of the speed vector 7' ( T ) with respect to the splitting $T_qM$ = R**N** ( q ) © $T_qS$* at q = Y ( T ) , it is equal to the component parallel to N ( X ) of the speed vector Y ( T ) with respect to the splitting

Tx ( TpM ) = R**N** ( X ) ©

TxS* at X = 7 ( T ) . Therefore ,

( s, ( t ) , NS ( 7 ( t )) = ( Y' ( t ) , n ( Y ( t )) = ( Y' ( t ) , GRAD F ( Y ( t ))) =
( F 0 Y ) ' ( t ) ,
( S' ( T ) , N ( S ( T )) dT = ( f 0 Y ) ' ( T ) dT = IIY ( b ) ll - llY ( a ) ll = r .

The proof also shows that the equality l ( y ) = r holds only for curves perpendicular to the spheres S* .

**Check your Progress 1**

Discuss Gauss—Bonnet Formula , Steiner's Formula

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Discuss Minkowski's Formula

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## 12.5 LET US SUM UP

In this unit we have discussed the definition and example of The Gauss—Bonnet Formula , Steiner's Formula , Minkowski's Formula

## 12.6 KEYWORDS

The Gauss—Bonnet Formula….. A surface M C R3 in R3 . Define a local orthonormal frame e1 , e2 , e3, over an open subset U of M

Steiner's Formula…. Steiner's formula , named after the Swiss geometer Jacob Steiner ( 1796 - 1863 ) asserts that for a compact convex subset K of Rn ,

Minkowski's Formula….. Being important geometrical invariants of a convex body , the coefficients of the polynomial in Steiner's formula are studied

## 12.7 QUESTIONS FOR REVIEW

Explain The Gauss—Bonnet Formula

Steiner's Formula

Minkowski's Formula

## 12.8 SUGGESTED READINGS

Differential Geometry, Differential Geometry & Application,
Introduction to Defferential Geometry, Basic of Differential Geometry.

## 12.9 ANSWERS TO CHECK YOUR PROGRESS

The Gauss—Bonnet Formula

Steiner's Formula

Minkowski's Formula

( answer for Check your Progress - 1 Q )

# UNIT-13 : THE PRINCIPAL AXIS THEOREM

**STRUCTURE**



## 13.0 OBJECTIVES

After studying this unit , you should be able to:

- Understand about The Principal Axis Theorem
- Induced Euclidean Structures on Tensor Spaces
- Curvature of Riemannian Manifolds
- Curvilinear Coordinates
- Equation of Straight Line

## 13.1 INTRODUCTION

Differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces Mathematical analysis of curves and surfaces had been developed to answer some of unanswered questions that appeared in calculus like the reasons for relationships between The Principal Axis Theorem , Induced

## 13.2 THE PRINCIPAL AXIS THEOREM

Sometimes a Euclidean linear space ( V , ( , ) ) is equipped with a second symmetric bilinear function { , }: V x V ^ R . In general , a symmetric bilinear function on a linear space can be defined by its matrix with respect to a basis or by its quadratic form . Using the inner product of the Euclidean structure it can also be defined with the help of a self - adjoint map .

If ( xi , . . . , xn ) is a basis of the vector space V and { , } is a bilinear function on V then the n x n matrix ( bij ) i<i , j<n with entries bij = {xi , xj} is called the matrix representation or simply the matrix of { , } with respect to the basis ( xi , . . . , xn ) . Fixing the basis we get a one to one correspondence between bilinear functions and n x n matrices . A bilinear form is symmetric if and only if its matrix with respect to a basis is symmetric .

Definition . The quadratic form of a bilinear function { , } is the function defined by the equality Q{ , } ( x ) = {x , x} . IS

Symmetric bilinear functions can be recovered from their quadratic forms with the help of the identity

{x y} = 1 ( Q{ , y ( x + y ) - Q{ , } ( x ) - Q{ , } ( y ) ) . ( i . 8 )

The following proposition establishes a bijection between bilinear functions on a Euclidean linear space V and linear endomorphisms L: V ^ V .

Proposition . Let ( V , ( , } ) be a Euclidean linear space .

Then for any linear endomorphism L: V ^ V , the map

{ , }l : V x V ^ R , {x , y}L = ( Lx , y}

is a bilinear function on V .

For any bilinear function { , } on V , there is a unique linear map L: V ^ V such that { , } = { , }L .

The bilinear function { , }l is symmetric if and only if L satisfies the identity ( Lx , y} = ( x , Ly} .

Proof . We prove only ( ii ) , the rest is obvious . Choose an orthonormal basis ( ei , . . . , en ) in V . Given a bilinear function { , } , the only possible choice for L ( x ) is

L ( x ) = ^2 ( Lx , ei}ei = ^{x , e^

= 1 i=1

which proves uniqueness . On the other hand , if we define L by the last equality , then

n n

( Lx , y} = <= {x , ej} ( ej , y} = {x , ^ ( e* , y}e^ = {x , y} ,

i=1 i=1

so L is a good choice . □

Definition . A linear endomorphism L of a Euclidean linear space is said to be self - adjoint ( with respect to the Euclidean structure ) if it satisfies the identity ( Lx , y} = ( x , Ly} .

Lemma . All the eigenvalues of a self - adjoint map L: V ^ V are real .

Proof . Let x+iy G C be an eigenvalue of L , v+iw G C ( g>V be a corresponding eigenvector . Then the real and imaginary part of the equation L ( v + iw ) = ( x + iy ) ( v + iw ) gives Lv = xv — yw and Lw = yv + xw . Since L is self - adjoint , we have

x ( v , w ) — y||w||2 = ( Lv , w ) = ( v , Lw ) = x ( v , w ) + y||v||2 ,

which yields y ( ||v||2 + ||w||2 ) =0 . However , v + iw = 0 because eigenvectors are non - zero vectors , therefore y = 0 . □

Theorem ( Principal axis theorem ) . Let V be a finite dimensional Euclidean linear space and let L: V ^ V be a self - adjoint linear transfor -

Notes

mation on V . Then there exists an orthonormal basis of V consisting of eigenvectors of L .

Proof . We prove by induction on the dimension n of V . The base case n =1 is trivial . Assume that it is true for n = k . Suppose n = k + 1 there exists a unit vector vi in V which is an eigenvector of L . Let W = v ( - = {w G V | v1 T w} . Then L ( W ) C W since we have

( Lw , v1 ) = ( w , Lv1 ) = ( w , A1v1 ) = A1 ( w , v1 ) = 0

for any w G W , where A1 is the eigenvalue belonging to v1 . Clearly L|W is self - adjoint . Since dim ( W ) = dim ( V ) — 1 = k , the induction assumption implies that there exists an orthonormal basis ( v2 , . . . , vn ) in W consisting of eigenvectors of L|W . But each eigenvector of L|W is an eigenvector of L , so ( v1 , . . . , vn ) is an orthonormal basis of V consisting of eigenvectors of L .

## 13.3 INDUCED EUCLIDEAN STRUCTURES ON TENSOR SPACES

Let ( V , ( , ) ) be a finite dimensional Euclidean linear space . Our goal now is to extend the dot product in a natural way to tensor spaces and exterior power spaces constructed from V . Consider the map

K: V2k = Vk x Vk ^ R ,

( ( v1 , . . . , vk ) , ( w1 , . . . , wk ) ) ^ ( v1 , w1 ) — ( vk , wk ) .

As this map is 2k - linear , it induces a linear map L: T ( 0>2k ) V ^ R such that K = L o ( g ) 2k . Composing L with the tensor product operation ( g>: T ( 0'k ) x T ( °>k ) ^ T ( 0>2k ) we obtain a bilinear function on T ( 0>k ) , which we also denote

by ( , ) .

Proposition . The induced bilinear function ( , ) on T ( 0>k ) V is sym - metric and positive definite . If ep . . . , en is an orthonormal basis of V , then eq ik ( 1 < i 1 , . . . , ik < n ) is an orthonormal basis of T ( 0>k ) V with respect to ( , ) " Proof . It is enough to show the second part of the proposition since orthonor - mal basis exists only for symmetric positive definite bilinear functions . This follows from

( eil . . . ifc , ejl . . . jk ) = L ( eil <g> ••• <g> eik <g> j <g> ••• <g> ejk )

= ( eii , eji ) • • • ( eik , ejk ) = ^il , jl • • • ^ifc , jk .

The linear space Ak ( V ) of alternating tensors is a linear subspace of T ( 0 , k ) , so the restriction of kf ( , ) defines a symmetric positive definite bilinear function on Ak ( V ) . Using the natural isomorphism ak between Ak ( V ) and Ak ( V ) we obtain a positive definite symmetric bilinear function ( , ) also on Ak ( V ) expressed by

( ^1 , ^2 ) = 1 ( «k ( ^i ) , ak ( ^2 ) ) Vwi , W2 € Ak ( V ) .

The reason why we divide by k! is to make the following proposition true . Proposition . If ei , . . . , en is an orthonormal basis of V , then

{eil A • • • A eik | 1 < ii < • • • < ik < n} is an orthonormal basis of Ak ( V ) .

<

Proof . If the increasing sequences 1 < i1 < ••• <ik < n and 1 < j1 < • jk < n are not the same , then no permutation of them can coincide , so

( eil A • • • A eik , ejl A • • • A ejk )

When the sequences are equal , then the permuted sequences in ( i ) , and jo - ( i ) , . . . , ja ( k ) coincide if and only if n = a , therefore

Let V* be the dual space of V . The inner product is a bilinear function , therefore it defines a linear map l: V ^ V* by 1 ( v ) ( w ) = ( v , w} . The linear map l is injective , since if l ( v ) = 0 , then 1 ( v ) ( v ) = ||v||2 = 0 , so v = 0 . This implies that l is a linear isomorphism between V and V* . The linear isomorphism l can be used to identify V and its dual space , and gives rise to a Euclidean linear space structure ( , }: V* x V* ^ R on V* , for which

( 1 ( v ) , 1 ( w ) } = ( v w} .

Proposition . If ( ei , . . . , en ) is an orthonormal basis of V , then its dual basis ( ei , . . . , en ) is an orthonormal basis of V* .

Proof . Since the basis is orthonormal , we have

1 ( ei ) ( ej ) = ( eA ej} = $ij = ei ( ej ^

thus , l ( ej ) = e - - > , and

( e\ ej } = ( 1 ( ei ) , 1 ( ej ) } = ( ei , ej} = Sij .

The Euclidean structure on V* induces a Euclidean structure on the tensor spaces T ( k>0 ) V and also on Ak ( V* ) . Finally , since I identifies V with its dual space , tensors of type ( k , l ) can be identified with tensors of type ( k + 1 , 0 ) and also with tensors of type ( 0 , k + l ) . Both identifications induces the same Euclidean structure on T ( k>l ) V , with respect to which the basis ej ) " 'j generated from any orthonormal basis of V will be an orthonormal basis in T ( k>l ) V .

Curvature

If V is an affine connection on a manifold M , then we may consider the operator

R ( X , Y ) = [Vx , Vy] — V[x , y]: X ( M ) ^ X ( M ) ,

where [VX , VY] = VX oVY — VY oVX is the usual commutator of opera - tors . The operator R ( X , Y ) , depending on X and Y , is called the curvature operator of the connection . The assignment

X ( M ) x X ( M ) x X ( M ) ^ X ( M )

( X , Y , Z ) ^ R ( X , Y ) ( Z )

is called the curvature tensor of the connection . To reduce the number of brackets , we shall denote R ( X , Y ) ( Z ) simply by R ( X , Y; Z ) . Thus , the letter R is used in two different meanings , later it will denote also a third mapping , but the number of arguments of R makes always clear which meaning is considered .

Proposition . The curvature tensor is linear over the ring of smooth functions in each of its arguments , and it is skew symmetric in the first two arguments .

Proof . Skew symmetry in the first two arguments is clear , since

R ( X , Y ) = [Vx , Vy] — V[x , y] = — [Vy , Vx] + V[Y , X] = —R ( Y , X ) .

According to this , it suffices to check linearity of the curvature tensor in the first and third arguments .

Linearity in the first argument is proved by the following identities .

R ( X1 + X2 , Y ) = [VXi+X2 , VY] — V[Xi +X2 , Y]

= [VXi + VX2 , VY] — V[Xi , Y] + [X2 , Y]

= [VXi , Y] + [VX2 , VY] — V[X1 , Y] — V[X2 , Y] = R ( Xi , Y ) + R ( X2 , Y ) .

and

R ( fX , Y; Z ) = ( [Vfx , Vy] — Vfx , Y] ) ( Z )

= fVX V Y Z — V Y ( f VX Z ) — V f [X , Y] - Y ( f ) X ( Z )

= f Vx Vy Z — f Vy Vx Z — Y ( f ) Vx Z — f V[x , y]Z + Y ( f ) Vx ( Z ) = f ( Vx Vy Z — Vy Vx Z — V[x , y]Z ) = fR ( X , Y; Z ) .

Additivity in the third argument is clear , since R ( X , Y ) is built up of the additive operators VX , VY and their compositions . To have linearity , we need

R ( X , Y; fZ ) = VxVy ( fZ ) - VrVx ( fZ ) - V[x , Y] ( fZ )

= Vx ( Y ( f ) Z + fVYZ ) - Vy ( X ( f ) Z + fVxZ )

[X , Y] ( f ) Z - f V[x , y]Z

= XY ( f ) Z + Y ( f ) Vx Z + X ( f ) Vy Z + f Vx Vy Z -

YX ( f ) Z - X ( f ) Vy Z - Y ( f ) Vx Z - f Vy Vx Z -

XY ( f ) Z + YX ( f ) Z - f V[x , y]Z

= f ( Vx Vy Z - Vy Vx Z - V[x , y]Z ) = fR ( X , Y; Z ) .

Proposition is interesting , because the curvature tensor is built up from covariant derivations , which are not linear operators over the ring of smooth functions .

We have already introduced tensor fields over a hypersurface . We can in - troduce tensor fields over a manifold in the same manner . A tensor field T of type ( k , l ) is an assignment to every point p of a manifold M a tensor T ( p ) of type ( k , l ) over the tangent space TpM . If dp . . . , dn are the basis vector fields defined by a chart over the domain of the chart , and we denote by dx1 ( p ) , . . . , dxn ( p ) the dual basis of d1 ( p ) , . . . , dn ( p ) , then a tensor field is uniquely determined over the domain of the chart by the components

( p ) = T ( p ) ( dF> , . . . , dF'; j , . . . , j ) .

We say that the tensor field is smooth , if for any chart from the atlas of M , the functions TW^ are smooth . We shall consider only smooth tensor fields . Tensor fields of type ( 0 , 1 ) are the vector fields , tensor fields of type ( 1 , 0 ) are the differential 1 - forms . Thus , a differential 1 - form assigns to every point of the manifold a linear function on the tangent space at that point . Differential 1 - forms form a module over the ring of smooth functions , which we denote by D1 ( M ) .

Every tensor field of type ( k , l ) defines an F ( M ) - multilinear mapping D1 ( M ) x ••• x D1 ( M ) x X ( M ) x ••• x X ( M ) ^ F ( M )

l times k times and conversely , every such F ( M ) - multilinear mapping comes from a tensor field . ( Check this! ) Therefore , tensor fields can be identified with F ( M ) - multilinear mappings D1 ( M ) x • • • x D1 ( M ) x X ( M ) x • • • x X ( M ) ^ F ( m ) . Tensor fields of type ( k , 1 ) , that is F ( M ) - multilinear mappings

D1 ( M ) x X ( M ) x ••• x X ( M ) ^ F ( M )

can be identified in a natural way with F ( M ) - multilinear mappings

X ( M ) x ••• x X ( M ) ^ X ( M ) .

By this identification , R: X ( M ) x ••• x X ( M ) ^ X ( M ) corresponds to R: Q1 ( M ) x X ( M ) x ••• x X ( M ) ^ F ( M ) , defined by R ( w;Xu . . . , Xk ) = w ( R ( Xi , . . . , Xfc ) ) .

Using these identifications , the curvature tensor is a tensor field of type ( 3 , 1 ) by Proposition . It is a remarkable consequence , that although the vec - tors VXZ ( p ) and VYZ ( p ) are not determined by the vectors X ( p ) , Y ( p ) , Z ( p ) , to compute the value of R ( X , Y; Z ) at p it suffices to know X ( p ) , Y ( p ) , Z ( p ) . Beside skew - symmetry in the first two arguments , the curvature tensor has many other symmetry properties .

Theorem ( First Bianchi Identity ) . If R is the curvature tensor of a torsion free connection , then

R ( X , Y; Z ) + R ( Y , Z; X ) + R ( Z , X; Y ) = 0

for any three vector fields X , Y , Z .

Proof . Let us introduce the following notation . If F ( X , Y , Z ) is a function of the variables X , Y , Z , then denote by ^0 F ( X , Y , Z ) or qxyz F ( X , Y , Z ) the sum of the values of F at all cyclic permutations of the variables ( X , Y , Z )

<= F ( X , Y , Z ) = F ( X , Y , Z ) + F ( Y , Z , X ) + F ( Z , X , Y ) .

o

We shall use several times that behind the cyclic summation ^o we may cyclically rotate X , Y , Z in any expression

<= F ( X , Y , Z ) = <= F ( Y , Z , X ) = <= F ( Z , X , Y ) .

ooo

The theorem claims vanishing of

<= R ( X , Y; Z )

o

= <= ( VxVyZ - VyVxZ - V[x , y]Z )

o

= <= ( VxVyZ - VxVzY - V[x , y]Z )

o

= <= ( Vx [Y , Z] - V[x , y]Z )

o

= <= ( Vz[X , Y] - V[x , y]Z ) = <=[Z , [X , Y]] ,

oo

but the last expression is 0 according to the Jacobi identity on the Lie bracket of vector fields . ( At the third and fifth equality we used the torsion free property of V . )

The presence of an affine connection on a manifold allows us to differentiate not only vector fields , but also tensor fields of any type .

Definition . Let ( M , V ) be a manifold with an affine connection . If w G H1 ( M ) is a differential 1 - form , X is a vector field , then we define the covariant derivative Vxw of w with respect to X to be the 1 - form

( Vxw ) ( Y ) = X ( w ( Y ) ) - w ( VxY ) , Y G X ( M ) .

In general , the covariant derivative VxT of a tensor field

T: Q} ( M ) x ••• x Q} ( M ) x X ( M ) x ••• x X ( M ) ^ F ( M )

of type ( k , l ) with respect to a vector field X is a tensor field of the same type , defined by

( VxT ) ( wi , . . . , wi; Xi , . . . , Xfc ) = X ( T ( wi , . . . , w , ; Xi , . . . , Xfc ) ) -

k

T ( w1 , . . . . , VX wi , . . . . , wl; X1 , . . . , Xk )

i=1 l

^ T ( wi , . . . , w , ; Xi , . . . , Vx Xj , . . . , Xk ) . ss j=1

For the case of the curvature tensor , this definition gives

( VxR ) ( Y , Z; W ) = Vx ( R ( Y , Z; W ) ) - R ( VxY , Z; W ) - R ( Y , VxZ; W ) - R ( Y , Z; Vx W ) .

Theorem ( Second Bianchi Identity ) . The curvature tensor of a torsion free connection satisfies

<= ( VxR ) ( Y , Z; W ) = 0 .

QXYZ

Proof . ( VxR ) ( Y , Z; W ) is the value of the operator Vx O R ( Y , Z ) - R ( VxY , Z ) - R ( Y , VxZ ) - R ( Y , Z ) OVx : X ( M ) ^ X ( M ) on the vector field W , hence we have to prove vanishing of the operator <= ( Vx O R ( Y , Z ) - R ( VxY , Z ) - R ( Y , VxZ ) - R ( Y , Z ) O Vx ) . ( 4 . 15 )

OXYZ

First , we have

y ( Vx o R ( Y , Z ) - R ( Y , Z ) oVx )

OXYZ

= y ( ( VxVyVz - VxVzVy - VxV[y , z] ) qxyz

( Vy Vz Vx - Vz Vy Vx - V[y , z]V x ) ) = y ( ( Vx Vy Vz - Vx Vz Vy - Vx V[y , z] )

QXYZ

( VXVYVZ - VXVZVY - V[Y , Z]VX ) ) = y ( V[Y , Z] VX - VX V[Y , Z] ) .

QXYZ

On the other hand ,

y ( - R ( VxY , Z ) - R ( Y , VxZ ) )

QXYZ

= y ( R ( VxZ , Y ) - R ( VxY , Z ) )

QXYZ

= y ( R ( VyX , Z ) - R ( VxY , Z ) )

QXYZ

= y R ( Vy X - Vx Y , Z ) = y R ( [Y , X ] , Z ) .

QXYZ QXYZ

Combining these results , operator ( 4 . 15 ) equals y ( V[y , z]Vx - VxV[Y , Z] + R ( [Y , X] , Z ) )

QXYZ

= y ( V[y , z]Vx - VxV[y , z]+ R[Z , Y] , X ) )

QXYZ

= Y ( V[Y , Z] VX - VXV[Y , Z] + V[Z , Y]VX - VXV[Z , Y] - V[[Z , Y] , X] )

QXYZ

= Y V[[y , z] , X] = VE0[[X , Y] , Z] =0 .

QXYZ

## 13.4 CURVATURE OF RIEMANNIAN MANIFOLDS

In the remaining part of this unit , we shall deal with Riemannian manifolds , therefore from now on assume that ( M , ( , } ) is a Riemannian manifold with Levi - Civita connection V and R is the curvature tensor of V .

Introduce the tensor R of type ( 4 , 0 ) , related to R by the equation

R ( X , Y; Z , W ) = {R ( X , Y; Z ) , W ) , which is called the Riemann - Christoffel curvature tensor of the Riemannian manifold . To simplify notation , we shall denote R also by R . This will not lead to confusion , since the Riemann - Christoffel tensor and the ordinary curvature tensor have different number of arguments .

Levi - Civita connections are connections of special type , so it is not surprising , that the curvature tensor of a Riemannian manifold has stronger symmetries than that of an arbitrary connection . Of course , the general results can be applied to Riemannian manifolds as well yielding

R ( X , Y; Z , W ) = - R ( Y , X; Z , W ) and ^ R ( X , Y; Z , W ) = 0 .

QXYZ

In addition to these symmetries , we have the following ones .

Theorem 4 . 9 . 5 . The Riemann - Christoffel curvature tensor is skew - symmetric in the last two arguments

R ( X , Y; Z , W ) = —R ( X , Y; W , Z ) .

Proof . By the compatibility of the connection with the metric , we have

X ( Y«Z , W ) ) ) = X«VyZ , W ) + ( Z , VyW ) )

= ( Vx Vy Z , W ) + ( VyZ , Vx W )

+ ( Vx Z , Vy W ) + ( Z , Vx Vy W ) ,

and similarly ,

Y ( X ( ( Z , W ) ) ) = ( VyVxZ , W )

+ ( Vx Z , Vy W ) + ( Vy Z , Vx W ) + ( Z , Vy Vx W ) .

We also have

[X , Y ] ( ( Z , W ) ) = ( V[x , y ] Z , W ) + ( Z , V[x , y]W ) .

Subtracting from the first equality the second and the third one and applying [X , Y] = X o Y — Y o X , we obtain

0 = ( Vx Vy Z — Vy Vx Z — V[x , y ]Z , W )

$+ ( Z , \nabla_X \nabla_Y W — \nabla_Y \nabla_X W — \nabla_{[X , Y]} W )$

$= R ( X , Y; Z , W ) + R ( X , Y; W , Z ) .$

For tensors having the symmetries of a Riemannian curvature tensor , we introduce the following

Definition . Let V be a finite dimensional linear space over R . An algebraic curvature tensor or Bianchi tensor over V is a 4 - linear map S: V x V x V x V ^ R satisfying the symmetry relations

$S ( X , Y , Z , W ) = - S ( Y , X , W , Z ) = —S ( X , Y , W , Z )$

and the Bianchi identity

$S ( X , Y , Z , W ) + S ( Y , Z , X , W ) + S ( Z , X , Y , W ) = 0 .$

An algebraic curvature tensor field or Bianchi tensor field over a manifold M is a tensor field of type ( 4 , 0 ) , which assigns to each point p of M a Bianchi tensor over TpM .

We know from linear algebra ( see equation ( 1 . 8 ) ) that a symmetric bilinear form is uniquely determined by its quadratic form . More generally , when a tensor has some symmetries , it can be reconstructed from its restriction to a suitable linear subspace of its domain . For algebraic curvature tensors , we have the following

Proposition . Let Si and S2 be algebraic curvature tensors ( or tensor fields ) . If Si ( X , Y; Y , X ) = S2 ( X , Y; Y , X ) for every X and Y , then Si = S2 .

Proof . The difference S = Si — S2 is also an algebraic tensor ( field ) , and S ( X , Y; Y , X ) = 0 for all X , Y . We have to show S = 0 .

We have for any X , Y , Z

$0 = S ( X , Y + Z; Y + Z , X )$

$= S ( X , Y; Y , X ) + S ( X , Y; Z , X ) + S ( X , Z; Y , X ) + S ( X , Z; Z , X ) = S ( X , Y; Z , X ) + S ( X , Z; Y , X ) + [S ( X , Y; Z , X ) + S ( Y , Z; X , X )$

" V '

=0

$+ S(Z, X; Y, X)] = 2S(X, Y; Z, X)$.

Now taking four arbitrary vectors ( vector fields ) X , Y , Z , W and using $S(X, Y; Z, X) = 0$, we obtain

$0 = S(X + W, Y; Z, X + W)$

$= S(X, Y; Z, X) + S(X, Y; Z, W) + S(W, Y; Z, X) + S(W, Y; Z, W) = S(X, Y; Z, W) + S(W, Y; Z, X)$,

i . e . , S is skew symmetric in the first and fourth variables . Thus ,

$S(X, Y; Z, W) = S(Y, X; W, Z) = - S(Z, X; W, Y) = S(Z, X; Y, W)$,

in other words , S is invariant under cyclic permutations of the first three vari - ables . But the sum of the three equal quantities $S(X, Y; Z, W)$, $S(Y, Z; X, W)$ and $S(Z, X; Y, W)$ is 0 because of the Bianchi symmetry , thus $S(X, Y; Z, W)$ is 0 .

Exercise Let S be an algebraic curvature tensor , and let $QS(X, Y) := S(X, Y; Y, X)$. Prove that $QS(X, Y) = QS(Y, X)$ and

$6S(X, Y; Z, W) = Qs(X + W, Y + Z) - Qs(Y + W, X + Z) + Qs(Y + W, X) - - Qs(X + W, Y) + Qs(Y + W, Z) - Qs(X + W, Z) + + Qs(X + Z, Y) - Qs(Y + Z, X) + Qs(X + Z, W) -$

$Qs(Y + Z, W) + Qs(X, Z) -$

$Qs(Y, Z) + Qs(Y, W) - Qs(X, W)$. Theorem . Assume that S is an algebraic curvature tensor . Then

$S(X, Y; Z, W) = S(Z, W; X, Y)$.

Proof . Label the vertices of an octahedron as shown in the figure . It follows for the upper two shaded triangles "Z" and "W" and subtract the identities corresponding to the two lower triangles "X" and "Y" , we obtain

$2S(X, Y; Z, W) - 2S(Z, W; X, Y) = 0$,

as we wanted to prove .

Corollary For the Riemann - Christoffel tensor , the identity R ( X , Y; Z , W ) = R ( Z , W; X , Y ) holds .

Definition . Let M be a Riemannian manifold , p a point on M , X and Y two non - parallel tangent vectors at p . The number

K ( X Y ) = R ( X , Y; Y , X )

K ( X , Y ) |X|21Y|2 - ( X , Y ) 2

is called the sectional curvature of M at p , in the direction of the plane spanned by the vectors X and Y in Tp M .

Proposition . If M is a Riemannian manifold with sectional curvature K , X and Y are linearly independent tangent vectors at p € M , a and ft are non - zero scalars , then

K ( X , Y ) = K ( X + Y , Y ) ;

K ( X , Y ) = K ( aX , ftY ) ;

K ( X , Y ) = K ( Y , X ) .

Furthermore , K depends only on the plane spanned by its arguments .

Proof . ( i ) follows from

R ( X + Y , Y; Y , X + Y ) = R ( X , Y; Y , X ) + R ( X , Y; Y , Y ) + R ( Y , Y; Y , X ) + R ( Y , Y; Y , Y ) = R ( X , Y; Y , X )

and

|X + Y|2|Y|2 - ( X + Y , Y ) 2

= ( |X|2 + |Y|2 + 2 ( X , Y ) ) |Y|2 - ( ( X , Y ) 2 + 2 ( X , Y ) |Y|2 + |Y|4 )
= |X|2|Y|2 - ( X , Y ) 2 .

( ii ) follows from

R ( aX , ftY; ftY , aX ) = a2ft2R ( X , Y; Y , X )

and

|aX|2|ftY|2 - ( aX , ftY ) 2 = a2ft2 ( |X|2|Y|2 - ( X , Y ) 2 ) .

( iii ) comes from the equalities R ( X , Y; Y , X ) = R ( Y , X; X , Y ) and |X|2|Y|2 — ( X , Y ) 2 = |Y|2|X|2 — ( Y , X ) 2 .

Finally , the last statement follows by ( i ) , ( ii ) , and ( iii ) as if xi , yi and x2 , y2 are two bases of a 2 - dimensional linear space , then we can transform one of them into the other by a finite number of elementary basis transformations of the form

( x , y ) ^ ( x + y , y ) ; ( x , y ) ^ ( ax , fty ) , where aft = 0; ( x , y ) ^ ( y , x ) .

Definition Riemannian manifolds , the sectional curvature function of which is constant , called spaces of constant curvature or simply space forms . A space form is elliptic or spherical if $K > 0$ , it is parabolic or Euclidean if $K = 0$ and is hyperbolic if $K < 0$ .

Typical examples are the n - dimensional sphere , Euclidean space and hyper - bolic space . Further examples can be obtained by factorization with fixed point free actions of discrete groups .

The following remarkable theorem resembles Theorem , but its proof is not so simple .

Theorem ( Schur ) . If M is a connected Riemannian manifold , dimM > 3 and the sectional curvature K ( Xp , Yp ) ( Xp , Yp G TpM ) depends only on p ( and does not depend on the plane spanned by Xp and Yp ) , then K is constant , that is , as a matter of fact , it does not depend on p either .

Proof . By the assumption ,

R ( X , Y; Y , X ) = f ( |X |2|Y |2 — ( X , Y ) 2 )

for some function f . Our goal is to show that f is constant . Consider the tensor field of type ( 4 , 0 ) defined by

S ( X , Y; Z , W ) = f ( ( X , W ) ( Y , Z ) — ( X , Z ) ( Y , W ) ) .

It is clear from the definition that S is skew - symmetric in the first and last two arguments . It has also the Bianchi symmetry — indeed ,

<= S ( X , Y; Z , W ) = <= f ( ( X , W ) ( Y , Z ) — ( X , Z ) ( Y , W ) )

OXYZ OXYZ

= <= f ( ( Y , W ) ( Z , x ) — ( X , Z ) ( Y , w ) ) = 0 ,

OXYZ

thus S is an algebraic curvature tensor field . We also have R ( X , Y; Y , X ) = S ( X , Y; Y , X ) , therefore R = S . Set

S ( X , Y; Z ) = f ( ( Y , Z ) X — ( X , Z ) Y ) .

Then for any vector field W , we have

( R ( X , Y; Z ) , W ) = R ( X , Y; Z , W ) = S ( X , Y; Z , W ) = ( S ( X , Y; Z ) , W ) , that is ,

R ( X , Y; Z ) = S ( X , Y; Z ) for all X , Y , Z .

Differentiating with respect to a vector field U we get

( Vu R ) ( X , Y; Z ) = ( Vu S ) ( X , Y; Z )

= Vu ( S ( X , Y; Z ) ) - S ( VuX , Y; Z )

- S ( X , VuY; Z ) - S ( X , Y; VuZ ) .

Since

Vu ( S ( X , Y; Z ) ) = U ( f ) ( ( Y , Z ) X - ( X , Z ) Y ) + f Vu ( ( Y , Z ) X - ( X , Z ) Y ) = U ( f ) ( ( Y , Z ) X - ( X , Z ) Y ) + f ( U ( Y , Z ) X + ( Y , Z ) VuX

- U ( X , Z ) Y - ( X , Z ) VuY )

= U ( f ) ( ( Y , Z ) X - ( X , Z ) Y ) + f ( ( VuY , Z ) X + ( Y , VuZ ) X+

+ ( Y , Z ) VuX - ( VuX , Z ) Y - ( X , VuZ ) Y - ( X , Z ) VuY ) = = U ( f ) ( ( Y , Z ) X - ( X , Z ) Y ) + S ( VuX , Y; Z ) + S ( X , VuY; Z ) + + S ( X , Y; VuZ ) ,

we obtain

( VuR ) ( X , Y; Z ) = ( VuS ) ( X , Y; Z ) = U ( f ) ( ( Y , Z ) X - ( X , Z ) Y ) .

Using the second Bianchi identity , this gives us

<= U ( f ) ( ( Y , Z ) X - ( X , Z ) Y ) = <= ( VuR ) ( X , Y; Z ) = 0 .

QUXY QUXY

If X € is an arbitrary tangent vector , then we can find non - zero vectors Y , Z = U € such that X , Y and U are orthogonal ( dimM > 3! ) . Then

0= <= U ( f ) ( ( Y , Z ) X - ( X , Z ) Y ) = X ( f ) ( U , U ) Y - Y ( f ) ( U , U ) X .

QUXY

Since X and Y are linearly independent and ( U , U ) is positive , X ( f ) = Y ( f ) = 0 follows , yielding that the derivative of f with respect to an ar - bitrary tangent vector X is 0 . This means that f is locally constant , and since M is connected , f is constant .

The curvature tensor is a complicated object containing a lot of information about the geometry of the manifold . There are some obvious ways to derive simpler tensor fields from the curvature tensor . Of course , simplicity is paid by losing information .

Definition . Let ( M , V ) be a manifold with an affine connection , R be the curvature tensor of V . The Ricci tensor Ric of the connection is a tensor field of type ( 2 , 0 ) assigning to the vector fields X and Y the function Ric ( X , Y ) the value of which at p G M is the trace of the linear mapping

TpM ^ TpM

Zp ^ R ( Zp , X ( p ) ; Y ( p ) ) , where Zp G TpM .

Proposition . The Ricci tensor of a Riemannian manifold is a symmetric tensor

Ric ( X , Y ) = Ric ( Y , X ) .

**Notes**

Proof . Let ei , . . . , en be an orthonormal basis in TpM , where p is an arbitrary point in the Riemannian manifold M . We can compute the trace of a linear mapping A: TpM ^ TpM by the formula

n

Tr A = <=<A ( ei ) , e*> .

i= 1

In particular ,

nn

Ric ( X , Y ) ( p ) = <=<R ( ei , X ( p ) ; Y ( p ) ) , ei> = <= R ( e* , X ( p ) ; Y ( p ) , ei )

i=1 i=1

nn

= <= R ( Y ( p ) , ei; ei , X ( p ) ) = <= R ( e* , Y ( p ) ; X ( p ) , ei )

i=1 i=1

= Ric ( Y , X ) ( p ) . □

Since the Ricci tensor of a Riemannian manifold is symmetric , it is uniquely determined by its quadratic form X ^ Ric ( X , X )

Definition . Let Xp G TpM be a non - zero tangent vector of a Riemannian manifold M . The Ricci curvature of M at p in the direction Xp is the number

r ( X4

|Xp|

curvature as follows

Ric ( Xp , Xp ) = ^ R ( ei , Xp; Xp , ei ) = ^ K ( X e )

r ( Xp ) = |Xp|2 = |Xp|2 = K ( Xp ei ) .

study the curvature tensor of a hypersurface M in Rn . According to the Levi - Civita connection V of a hypersurface can be expressed as V = P o d , where d is the derivation rule of vector fields along the hypersurface as defined in Definition , P is the orthogonal projection of a tangent vector of Rn at a hypersurface point onto the tangent space of the hypersurface at that point . Comparing Definition to formula , we see that the derivation d of vector fields along a hypersurface is induced by the Levi - Civita connection of Rn , which we also denoted by d . As the curvature of Rn is 0 ,

dx o dy — dy o dx = d[x , Y]

holds for any tangential vector fields X , Y G X ( M ) .

We have

VxVyZ = P ( dx VyZ ) = P ( dx ( dyZ - {dyZ , N ) N ) )

= P ( dxdyZ ) - P ( X ( {dyZ , N ) ) N ) - P ( {dyZ , N ) dxN ) = P ( dxdyZ ) - {dyZ , N ) dxN , where X , Y , Z e X ( M ) . Similarly ,

Vy VxZ = P ( dydxZ ) - {dxZ , N ) dyN .

Combining these equalities with

v[x , y]z = P ( d[x , y ]Z )

we get the following expression for the curvature tensor R of M

R ( X , Y; Z ) = ( Vx Vy Z - Vy Vx Z ) - V[x , y]Z =

= P ( ( dx dy Z - dy dx Z ) - d[x , y]Z ) - {dyZ , N ) dxN + {dxZ , N ) dyN = {dxZ , N ) dyN - {dyZ , N ) dxN .

Since {Z , N ) is constant zero ,

0 = X ( {Z , N ) ) = {dx Z , N ) + {Z , dx N )

and

0 = Y ( {Z , N ) ) = {dy Z , N ) + {Z , dy N ) .

Putting these equalities together we deduce that

R ( X , Y; Z ) = {Z , dyN ) dxN - {Z , dxN ) dyN = {Z , L ( Y ) ) L ( X ) - {Z , L ( X ) ) L ( Y ) .

Comparing the formula

R ( X , Y; Z ) = {Z , L ( Y ) ) L ( X ) - {Z , L ( X ) ) L ( Y )

relating the curvature tensor to the Weingarten map on a hypersurface with Gauss' equations we see that the curvature tensor R coincides with the curva - ture tensor defined there . This way , the last equation can also be considered as a coordinate free display of Gauss' equations

## 13.5 CURVILINEAR COORDINATES

In geometry , **curvilinear coordinates** are a coordinate system for Euclidean space in which the coordinate lines may be curved . These coordinates may be derived from a set of Cartesian coordinates by using a transformation that is locally invertible ( a one - to - one map ) at each point . This means that one can convert a point given in a Cartesian coordinate system to its curvilinear coordinates and back . The name curvilinear coordinates , coined by the French mathematician Lamé , derives from the fact that the coordinate surfaces of the curvilinear systems are curved .

Well - known examples of curvilinear coordinate systems in three - dimensional Euclidean space ( $\mathbf{R}^3$ ) are cylindrical and spherical polar coordinates . A Cartesian coordinate surface in this space is a coordinate plane; for example z = 0 defines the x - y plane . In the same space , the coordinate surface r = 1 in spherical polar coordinates is the surface of a unit sphere , which is curved . The formalism of curvilinear coordinates provides a unified and general description of the standard coordinate systems .

Curvilinear coordinates are often used to define the location or distribution of physical quantities which may be , for example , scalars , vectors , or tensors . Mathematical expressions involving these quantities in vector calculus and tensor analysis ( such as the gradient , divergence , curl , and Laplacian ) can be transformed from one coordinate system to another , according to transformation rules for scalars , vectors , and tensors . Such expressions then become valid for any curvilinear coordinate system .

Depending on the application , a curvilinear coordinate system may be simpler to use than the Cartesian coordinate system . For instance , a physical problem with spherical symmetry defined in $\mathbf{R}^3$ ( for example , motion of particles under the influence of central forces ) is usually easier to solve in spherical polar coordinates than in Cartesian coordinates . Equations with boundary conditions that follow coordinate surfaces for a particular curvilinear coordinate system may be easier to solve in that system . One would for instance describe the motion of a particle in a rectangular box in Cartesian coordinates , whereas one would prefer spherical coordinates for a particle in a sphere . Spherical coordinates are one of the most used curvilinear coordinate systems in such fields as Earth sciences , cartography , and physics ( in particular quantum mechanics , relativity ) , and engineering .

using Cartesian coordinates ( x , y , z ) [equivalently written ( $x^1$ , $x^2$ , $x^3$ ) ] , by , where $\mathbf{e}_x$ , $\mathbf{e}_y$ , $\mathbf{e}_z$ are the standard basis vectors .

It can also be defined by its **curvilinear coordinates** ( $q^1$ , $q^2$ , $q^3$ ) if this triplet of numbers defines a single point in an unambiguous way . The relation between the coordinates is then given by the invertible transformation functions:

The surfaces $q^1$ = constant , $q^2$ = constant , $q^3$ = constant are called the **coordinate surfaces**; and the space curves formed by their intersection in pairs are called the **coordinate curves** . The **coordinate axes** are determined by the tangents to the coordinate curves at the intersection of three surfaces . They are not in general fixed directions in space , which happens to be the case for simple Cartesian coordinates , and thus there is generally no natural global basis for curvilinear coordinates .

In the Cartesian system , the standard basis vectors can be derived from the derivative of the location of point P with respect to the local

coordinate

Applying the same derivatives to the curvilinear system locally at point P defines the natural basis vectors: Such a basis , whose vectors change their direction and / or magnitude from point to point is called a **local basis** . All bases associated with curvilinear coordinates are necessarily local . Basis vectors that are the same at all points are **global bases** , and can be associated only with linear or affine coordinate systems .

Note: for this article **e** is reserved for the standard basis ( Cartesian ) and **h** or **b** is for the curvilinear basis . These may not have unit length , and may also not be orthogonal . In the case that they are orthogonal at all points where the derivatives are well defined , we define the Lamé coefficients ( after Gabriel Lamé ) by and the curvilinear orthonormal basis vectors by It is important to note that these basis vectors may well depend upon the position of P; it is therefore necessary that they are not assumed to be constant over a region . ( They technically form a basis for the tangent bundle of at P , and so are local to P . ) In general , curvilinear coordinates allow the natural basis vectors $\mathbf{h}_i$ not all mutually perpendicular to each other , and not required to be of unit length: they can be of arbitrary magnitude and direction . The use of an orthogonal basis makes vector manipulations simpler than for non - orthogonal . However , some areas of physics and engineering , particularly fluid mechanics and continuum mechanics , require non - orthogonal bases to describe deformations and fluid transport to account for complicated directional dependences of physical quantities .

Curvilinear coordinate systems in three - dimensional Euclidean space ( $\mathbf{R}^3$ ) are cylindrical and spherical polar coordinates . A Cartesian coordinate surface in this space is a coordinate plane; for example z = 0 defines the x - y plane . In the same space , the coordinate surface r = 1 in spherical polar coordinates is the surface of a unit sphere , which is curved . The formalism of curvilinear coordinates provides a unified and general description of the standard coordinate systems .

Curvilinear coordinates are often used to define the location or distribution of physical quantities which may be , for example , scalars , vectors , or tensors . Mathematical expressions involving these quantities in vector calculus and tensor analysis ( such as the gradient , divergence , curl , and Laplacian ) can be transformed from one coordinate system to another , according to transformation rules for scalars , vectors , and tensors . Such expressions then become valid for any curvilinear coordinate system

## 13.6 EQUATION OF STRAIGHT LINE

General equation

A straight line is defined by a linear equation whose general form is

$$Ax + By + C = 0 ,$$

where A , B are not both 0 .

The coefficients A and B in the general equation are the components of vector $\mathbf{n} = ( A , B )$ normal to the line . The pair $\mathbf{r} = ( x , y )$ can be looked at in two ways: as a point or as a radius - vector joining the origin to that point . The latter interpretation shows that a straight line is the locus of points $\mathbf{r}$ with the property

$$\mathbf{r}\cdot\mathbf{n} = \text{const} .$$

That is a straight line is a locus of points whose radius - vector has a fixed scalar product with a given vector $\mathbf{n}$ , normal to the line . To see why the line is normal to $\mathbf{n}$ , take two distinct but otherwise arbitrary points $\mathbf{r_1}$ and $\mathbf{r_2}$ on the line , so that

$$\mathbf{r_1}\cdot\mathbf{n} = \mathbf{r_2}\cdot\mathbf{n} .$$

But then we conclude that

$$( \mathbf{r_1} - \mathbf{r_2} ) \cdot\mathbf{n} = 0 .$$

In other words the vector $\mathbf{r_1} - \mathbf{r_2}$ that joins the two points and thus lies on the line is perpendicular to $\mathbf{n}$ .

**Notes**

Normalized equation

The norm $\|\mathbf{n}\|$ of a vector $\mathbf{n} = ( A , B )$ is defined via $\|\mathbf{n}\|^2 = A^2 + B^2$ and has the property that , for any non - trivial vector **n** , $\mathbf{n} / \|\mathbf{n}\|$ is a unit vector , i . e . , $\| \mathbf{n} / \|\mathbf{n}\| \| = 1$ .

Note that the line defined by a general equation would not change if the equation were to be multiplied by a non - zero coefficient . This property can be used to keep the coefficient A non - negative . It can also be used to normalize the equation by dividing it by $\|\mathbf{n}\|$ . As a result , in a normalized equation

$Ax + By + C = 0$ ,

$A^2 + B^2 = 1$ .

( In the applet , the coefficients of the normalized equation are rounded to up to 6 digits , for which reason the above identity may only hold approximately . )

The normalized equation is conveniently used in determining the distance from a point to a line .

Intercept - intercept

Assume a straight line intersects x - axis at ( a , 0 ) and y - axis at ( 0 , b ) . Then it is defined by the equation

$$x / a + y / b = 1 ,$$

which also can be written as

$$xb + ya = ab .$$

The latter form is somewhat more general as it allows either a or b to be 0 . a and b are defined as x - intercept and y - intercept of the linear function . These are signed distances from the points of intersection of the line with the axes .

Point - slope

The equation of a straight line through point ( a , b ) with a given slope of m is

$y = m(x - a) + b$, or $y - b = m(x - a)$.

As a particular case , we have

Slope - intercept equation

The equation of a line with a given slope m and the y - intercept b is

$y = mx + b$.

This is obtained from the point - slope equation by setting a = 0 . It must be understood that the point - slope equation can be written for any point on the line , meaning that the equation in this form is not unique . The slope - intercept equation is unique because if the uniqueness for the line of the two parameters: slope and y - intercept .

Parametric equation

A line through point $\mathbf{r_0} = (a, b)$ parallel to vector $\mathbf{u} = (u, v)$ is given by

$(x, y) = (a, b) + t\cdot(u, v)$,

where t is any real number . In the vector form , we have

$\mathbf{r} = \mathbf{r_0} + t\cdot\mathbf{u}$,

where $\mathbf{r} = (x, y)$.

Implicit equation

A line through point $\mathbf{r_0} = (a, b)$ perpendicular to vector $\mathbf{n} = (m, n)$ is given by

$m(x - a) + n(y - b) = 0$,

or if we take $\mathbf{r} = (x, y)$, a generic point on the line , we see that

$\mathbf{n}\cdot(\mathbf{r} - \mathbf{r_0}) = 0$,

where dots indicates the scalar product of two vectors .

Check your Progress 1

Discuss The Principal Axis Theorem

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Discuss Curvilinear Coordinates

## 13.7 LET US SUM UP

In this unit we have discussed the definition and example of The Principal Axis Theorem , Induced Euclidean Structures on Tensor Spaces , Curvature of Riemannian Manifolds , Curvilinear Coordinates , Equation of Straight Line

## 13.8 KEYWORDS

The Principal Axis Theorem ….Euclidean linear space ( V , ( , ) ) is equipped with a second symmetric bilinear function { , }: V x V ^ R

Induced Euclidean Structures on Tensor Spaces …. Let ( V , ( , ) ) be a finite dimensional Euclidean linear space

Curvature of Riemannian Manifolds ….. In the remaining part of this unit , we shall deal with Riemannian manifolds

Curvilinear Coordinates ….. In geometry , **curvilinear coordinates** are a coordinate system for Euclidean space in which the coordinate lines may be curved

Equation of Straight Line …..A straight line is defined by a linear equation whose general form is Ax + By + C = 0 ,

## 13.9 QUESTIONS FOR REVIEW

Explain The Principal Axis Theorem

## 13.10 SUGGESTED READINGS

Differential Geometry, Differential Geometry & Application,

Introduction to Defferential Geometry, Basic of Differential Geometry.

## 13.10 ANSWERS TO CHECK YOUR PROGRESS

The Principal Axis Theorem

Curvilinear Coordinates

( answer for Check your Progress - 1 Q )

# UNIT-14 : THE FRENET - SERRET FORMULAS

**STRUCTURE**



## 14.0 OBJECTIVES

After studying this unit , you should be able to:

- Understand about The Frenet –
- Serret Formulas
- Parallel Translation ,
- Gauss Map , Helix , Normal Forms

## 14.1 INTRODUCTION

Differential geometry arose and developed as a result of and in connection to the mathematical analysis of curves and surfaces Mathematical analysis of curves and surfaces had been developed to answer some of unanswered questions that appeared in calculus like the reasons for relationships between The Frenet - Serret Formulas , Parallel Translation , Gauss Map , Helix , Normal Forms

# 14.2 THE FRENET - SERRET FORMULAS

The Frenet - Serret frame of a space curve

We will consider smooth curves given by a parametric equation in a three - dimensional space . That is , writing bold - face letters of vectors in three dimension , a curve is described as r = f ( t ) , where F' is continuous in some interval I; here the prime indicates derivative . The length of such a curve between parameter values $t_0$ G I and ti G I can be described as

a ( tC= / |F' ( t ) |dt = /

J to J to

where , for a vector u we denote by |u| its length; here we assume $t_0$ is fixed and ti is variable , so we only indicated the dependence of the arc length on t\ . Clearly , a is an increasing continuous function , so it has an inverse $a^{-1}$; it is customary to write s = a ( t ) . The equation

$$r = F ( a^{-1} ( s ) ) \quad s \in J == \{a ( t ) : t \in I\}$$

is called the re - parametrization of the curve r = F ( t ) ( t G I ) with respect to arc length . It is clear that J is an interval . To simplify the description , we will always assume that r = F ( t ) and s = a ( t ) , so we will just use the variables r , t , and s instead of using function notation . We will use prime to indicate the differentiation d / dt , while the differentiation d / ds will not be abbreviated . We will assume that r' =0 for any t G I; then r' is a tangent vector to the curve corresponding to the given parameter value t . [11]

By the fundamental Theorem of Calculus , equation implies

s' = ds = df ) = |F' ( t ) |Hr1 .

Hence , by the chain rule of differentiation we have

Notes

d dt d 1 d 1 d

ii ,

ds ds dt \ds / dt J dt |r'| dt

where the second equation follows from equation by the Fundamental Theorem of calculus . When using differential operator notation as in d / ds , everything after the differential operator up to the next + or — sign needs to be differentiated; the expression preceding the differential operator is not to be differentiated . The unit tangent vector T is defined as

T =f — r' = dr; T = |r'|r = ds;

We will assume that k = 0 . The unit normal vector is defined as

N == 1 dT .

k ds

Note that $|T|^2$ = T • T , so by the product rule of differentiation ,

dT 1 dT dT

"dT • = 2 ( ~d^ • + • ~ds ) = ;

hence T is perpendicular to dT / ds , and so N is perpendicular to T . The unit binormal vector is defined as

B =f T x N .

The vectors T , N , B form the basic unit vectors of a coordinate system especially useful for describing the the local properties of the curve at the given point . These three vectors form what is called the Frenet - Serret frame . Equation implies that the vectors T , N , B form a right handed system of pairwise perpendicular unit vectors . Any cyclic permutation of these vectors also form a right handed system of pairwise perpendicular unit vectors; therefore we have

T x N = B ,

N x B = T , B x T = N .

If the equation **r** = **F** ( t ) describes a moving point , where t is time , then **r** is the velocity vector of the moving point at time t . That is , the length of **r'** is its speed , while the direction of r' is its direction of its movement . If **r'** = 0 , then the point stopped moving at the given time , and then it may resume its movement in a different direction . This means that even though the function **F** is differentiable , the tangent line to the curve described by the function may not be defined at this point .

A unit vector is a vector of length 1 .

We will comment later on what happens when k = 0 .

The Frenet—Serret formulas

As |N| = 1 , we have $|N|^2$ = N • N = 1 , and so , similarly to equation ( 8 ) , we have

dN ~

•

N

=

0

.

ds

That is dN / ds is perpendicular to N , so we have

= aT + t B

ds

for some numbers a and t ( depending on t ) . Here t is called the torsion of the curve at the point; the value of a will be determined below . Using this equation , equations , and the product rule of differentiation for vector products , we have dB = d ( T x N ) = dT dN

ds ds ds ds

= kN x N + T x ( aT + tB ) = tT x B = —tN;

the third equation follows by equations . The first term of the third member is zero; so is first term after distributing the cross product in the second term in the third member . Hence , using equation , we have

dN = d ( B x T ) = dB dT

ds ds ds ds

= —tN x T + B x ( kN ) = tB + kB x N = —kT + tB;

This equation shows that a in equation equals —k; however , this fact and equation itself is no longer of any interest , since this equation is subsumed in the last equation . Equations are called the Frenet - Serret formulas . To summarize these formulas , we have

dT

—— = kN , ds

dN

—— = —kT + t B ,

ds

dB

— = —t N . ds

The first three derivatives of r

As we mentioned above , we will indicate derivation with respect to t by prime . According to equations

r' = |r'|T = s'T .

We can do further differentiations with respect to t by using equations . We have

r" = s" T + s'T' = s" T + ( s' ) $^2$ ^ = s"T + ( s' ) $^2$kN . ds

Further , repeatedly using equations , we have

= s'''T + s'' T' + ( 2s's''k + ( s' ) $^2$k' ) N + ( s' ) $^2$kN' , dT , , ,

, , o ^ ' 3 dN

= s '''T + s ''s '-------------------------- + ( 2s ' s'' K + ( s ' ) $^2$k ' ) N + ( s ' )

$^3$K --------------------------------------- ds v----------------------------------------------v 2 , v !

ds

= s '''T + s ''s 'kN + ( 2s's ''k + ( s ' ) $^2$k ' ) N + ( s ' ) $^3$k ( —kT + t B )

= ( s ''' — ( s ' ) $^3$k ) T + ( 3s''s'k + ( s ' ) $^2$k ' ) N + ( s ' ) $^3$ kt B .

It is now easy to express k and t in terms of derivatives with respect to t

r' x r'' = ( s ' ) $^3$kB , so using equation we obtain that K =

by noting that |B| = 1 . By equations we have

( r' x r'' ) • r''' = ( s ' ) $^6$k$^2$t .

Hence , using equation and noting that |B| = 1 , we obtain that

( r ' x r '' ) • r '''

|r x r

Examples and discussion

Since the curvature k and the torsion t are defined in terms of the local coordinate frame T , N , B in arc - length parametrization , they only depend on the shape of the curve and not on the choice of the coordinate system x , y , z and the choice of the parameter t . Hence , for a curve that we want to calculate the curvature or the torsion , we may set up the coordinate system x , y , z and choose a parametrization that make these calculations especially simple .

The curvature of a circle

We consider the circle of radius R > 0 lying in the x , y plane , and centered at the origin . This circle can be parametrized by the equation

r = R ( i cos t + j sin t ) ,

where i , j , and k are the unit coordinate vectors in the directions of the positive x , y , and z axes , respectively . We have

r ' = R ( —i sin t + j cos t ) and

= —R ( i cos t + j sin t ) .

Hence

i j k

—R sin t R cos t 0 —R cos t —R sin t 0

We also have

$|r'| = rV \cos^2 1$

: + sin t = R .

Hence equation gives that

$R^2$ 1

$\kappa = R^3 = R$ .

Thus , the curvature of a circle is the reciprocal of the radius . For this reason , given any curve , 1 / k is called the radius of curvature; it is the radius of the osculating circle: given an curve by the equation r = F ( t ) , the osculating circle at a point corresponding to the parameter value $t = t_o$ ( i . e . , at the point with position vector $r_o$ = F ( to ) ) is a circle with equation r = G ( t ) at the for which

G ( to ) = F ( to ) , G' ( to ) = F' ( to ) , and G" ( to ) = F" ( to ) .

For the existence of such a circle , one needs to assume that F' ( $t_o$ ) = 0 and F" ( $t_o$ ) = 0 . The case that F' ( to ) = 0 is a case of bad parametrization , when the curve may or may not have a tangent line and a curvature , but the equation is not suitable for determining the tangent line or the curvature . In this case , one needs to re - parametrize the curve in such a way that the derivative at the point with position vector F ( $t_o$ ) is not zero .[3] If F' ( $t_o$ ) = 0 but F" ( $t_o$ ) = 0 , the curvature is 0 , and the osculating circle degenerates into a straight line; in fact , the tangent line can be considered the osculating "circle" in this case , and one may say that the corresponding radius of curvature is infinite .

The curvature and the torsion of a helix

A helix in the standard position can be described by the equation

r = i Rcos t + j R sin t + ct k ( R > 0 ) .

We have

—i R sin t + j R cos t + c k , —i R cos t — j R sin t , i R sin t — j R cos t .

Therefore

$|r'| = \backslash JR^2 ( \sin^2 t + \cos^2 t ) + c^2 = t / R^2 + c^2$ .

Furthermore ,

$i\ cR \sin t — j\ cR \cos t + R^2 ( \sin^2 1 + \cos^2 1 ) k$

$= i\ cR \sin t — j\ cR \cos t + R^2 k$ .

$|r' \times r''| = \backslash Jc^2R^2 ( \cos^2 t + \sin^2 t ) + R^4 = R\backslash / c^2 + R^2$

And

$( r' \times r'' ) \bullet r''' = R\hat{}c^2 + R^2 = cR^2 \sin^2 t + cR^2 \cos^2 t = cR^2$ .

Hence , using formula we obtain

$R\hat{}c^2 + R^2\ R$

$^K ( R^2 + c^2 )^{3/2}\ R^2 + c^2$ .

Similarly , using formula

$^T R^2 ( c^2 + R^2 )$

An osculating curve to a given curve r = F ( t ) is a curve r = G ( t ) satisfying equations; these equations can also be described by saying that the curves r = F ( t ) and r = G ( t ) have a second order contact at the given point . One can generalize this to an arbitrary integer n by saying that the curve r = F ( t ) and r = F ( t ) have an order n contact for a given parameter value $t = t_0$ if F' ( to ) = 0 and

$G^{(k)}( t_0 ) = F^{(k)}( t_0 )$ for all k with $0 < k < n$ .

The osculating plane is the plane spanned by the vectors T and N . The osculating circle lies in this place . The osculating plane has a second order contact with the curve at the point given by a parameter value $t = t_0$ . More generally , if $F' ( t_0 ) = 0$ and $F'' ( t_0 ) = 0$ , then any plane curve that has a second order contact with the curve r = F ( t ) at the parameter value $t = t_0$ lies entirely in the osculating plane . If $F'' ( t_0 ) = 0$ then the osculating plane is not determined since k = 0 in this case so the vector N is not determined . The torsion expresses the speed with which the osculating place turns as the arc - length parameter changes since B is normal to the osculating plane

The derivation of the Frenet - Serret formulas shows the theoretical usefulness of arc - length parametrization . Re - parametrizing a curve with respect to arc - length is rarely done in practice , since the integrals involved cannot usually be evaluated , and a more useful procedure is to rewrite the formulas derived with arc - length parametrization in terms of the original parameter , as was done in formulas

**Notes**

Every vector can be expressed as a linear combination of the basic unit vectors **T , N , B;** as dN / ds is perpendicular to **N ,** the coefficient of **N** in this linear combination is 0 .

## 14.3 PARALLEL TRANSLATION

Let M be a manifold with a connection , and 7: I ^ M be an immersed curve . Then we say that a vector field X e X ( 7 ) is parallel along 7 if

Dy X = 0 .

Thus , in this terminology , 7 is a geodesic if its velocity vector field is parallel . Further note that if M is a submanifold of $R^n$ , the , by the earlier results in this section , X is parallel along 7 if and only $( X' )^T = 0$ .

Example Let M be a two dimensional manifold immersed in $R^n$ , 7: I ^ M be a geodesic of M , and X e $X_M$ ( 7 ) be a vector field along 7 in M . Then X is parallel along 7 if and only if X has constant length and the angle between X ( t ) and 7' ( t ) is constant as well . To see this note that $( 7'' )^T = 0$ since 7 is a geodesic; therefore ,

( X , 7' ) ' = ( X ' , 7'} + <X , $_7$'' ) = ( X ' , 7' ) .

So , if $( X' )^T = 0$ , then it follows that ( X , 7' ) is constant which since 7' and X have both constant lengths , implies that the angle between X and 7' is constant . Conversely , suppose that X has constant length and makes a constant angle with 7' . Then ( X , 7' ) is constant , and the displayed expression above implies that ( X , 7' ) = 0 is constant . Furthermore , 0 = ( X , X ) ' = 2<X , X' ) . So X' ( t ) is orthogonal to both X ( t ) and Y' ( t ) . If X ( t ) and q' ( t ) are linearly dependent , then this implies that X' ( t ) is orthogonal to $T_{Y ( t )}$ M , i . e . , $( X' )^T = 0$ . If X ( t ) and q' ( t ) are linearly dependent , then $( X' )^T = D_7 ( X ) = D_7 ( / {}_7' ) = / D_7 ( 7' ) = 0$ .

Example ( Foucault's Pendulum ) . Here we explicitly compute the parallel translation of a vector along a meridian of the sphere . To this end let

$X(0,0) := (\cos(0)\sin(0), \sin(0)\sin(0), \cos(0))$

be the standard parametrization or local coordinates for $S^2$ — { ( 0 , 0 , ±1 ) } . Suppose that we want to parallel transport a given unit vector $V_0$ e $T_X$ ( $\$_0$ , $\wedge_0$ ) $S^2$ along the meridian X ( 0 , $0_o$ ) , where we identify tangent space of $S^2$ with subspaces of $R^3$ . So we need to find a mapping V: [0 , 2n] ^ $S^2$ such that V ( 0 ) = $V_0$ and V' ( 0 ) T $T_x$ ( 07o ) $S^2$ . The latter condition is equivalent to the requirement that

$$V'(0) = A(0) X(0, 0o),$$

since the normal to $S^2$ at the point X ( 0 , 0 ) is just X ( 0 , 0 ) itself . To solve the above differential equation , let

El ( 0 ) := = ( — sin ( 0 ) , cos ( 0 ) _ 0 ) ,

( cos ( 9 ) cos ( $\wedge_Q$ ) , sin ( 9 ) cos ( $\wedge_Q$ ) , — sin ( $0_Q$ ) ) .

Now note that ( Ei ( 9 ) , $E_2$ ( 9 ) } forms an orthonormal basis for $T_X$ ( $e_0$ , $\wedge_0$ ) $S^2$ . Thus ( 2 ) is equivalent to

( V' ( 9 ) , E1 ( 9 ) ) = 0 and ( V' ( 9 ) , E2 ( 9 ) ) = 0 .

So it remains to solve this differential equation . To this end first recall that since $V_Q$ has unit length , and parallel translation preserves length , we may write

$V(9) = \cos(a(9)) E_1(9) + \sin(a(9)) E_2(9)$ .

So differentiation yields that

$V' = El \cos(a) - \sin(a) a'E_1 + \sin(a) E2 + \cos(a) a'E_2$ .

Further , it is easy to compute that

$El = -\cos(\wedge_Q) E_2 - \sin(\wedge_Q) E_3$ and $E2 = \cos(0_Q) E_1$ ,

where $E_3(9) := X(9, 0_Q)$ . Thus we obtain:

$V' = \sin(a)(\cos(\wedge_Q) - a') E_1 + \cos(a)(a' - \cos(\wedge_Q)) E_2 + (*) E_3$ .

So for to be satisfied , we must have $a' = \cos(\wedge_Q)$ or

$a(9) = \cos(\wedge_Q) t + a(0)$ ,

which in turns determines V . Note in particular that the total rotation of

V with respect to the meridian X ( 9 , <$f_Q$ ) is given by

$$\alpha(2\pi) - \alpha(0) - \int_0^{2\pi} \alpha' d\theta - 2\pi \cos(\phi_0).$$

Thus

$$\phi_0 = \cos^{-1}\left(\frac{\alpha(2\pi) - \alpha(0)}{2\pi}\right).$$

The last equation gives the relation between the precession of the swing plane of a pendulum during a 24 hour period , and the longitude of the location of that pendulum on earth , as first observed by the French Physicist Leon Foucault in 1851 .

L e m m a 0 . 1 4 . Let I c R and U c $R^n$ be open subsets and F: I x U ^ $R^n$ , be $C^1$ . Then for every $t_Q$ <= I and $x_Q$ <= U there exists an e > such that for every 0 < e < e there is a unique curve x: ( $t_Q$ — e , $t_Q$ + e ) ^ $R^n$ with x ( $t_Q$ ) = $x_Q$ and x' ( t ) = F ( t , x ( t ) ) .

Proof . Define F: I x U ^ $R^{n+1}$ by F ( t , x ) := ( 1 , F ( t , x ) ) . Then , by Theorem there exists an e > 0 and a unique curve x: ( $t_0$—e , $t_0$+e ) ^ $R^{n+1}$ , for every 0 < e < e , such that x ( $t_0$ ) = ( 1 , $x_0$ ) and x' ( t ) = F ( x ( t ) ) . It follows then that x ( t ) = ( t , x ( t ) ) , for some unique curve x: ( $t_0$ — e , $t_0$ + e ) ^ $R^n$ . Thus F ( x ( t ) ) = ( 1 , F ( t , x ( t ) ) ) , and it follows that x' ( t ) = F ( t , x ( t ) ) .

Lemma . Let A ( t ) , t G I , be a $C^1$ one - parameter family of matrices . Then for every $x_0$ G $R^n$ and $t_0$ G I , there exists a unique curve x: I ^ $R^n$ with x ( $t_0$ ) = $x_0$ such that x' ( t ) = A ( t ) ■ x ( t ) .

Proof . Define F: I x $R^n$ ^ $R^n$ by $F_t$ ( x ) = A ( t ) ■ x . there exists a unique curve x: ( $t_0$ — e , $t_0$ + e ) ^ $R^n$ with x ( $t_0$ ) = $x_0$ such that $F_t$ ( x ( t ) ) = x' ( t ) for all t G ( $t_0$ — e , $t_0$ + e ) .

Now let J C I be the union of all open intervals in I which contains $t_0$ and such that x' ( t ) = F ( x ( t ) ) for all t in those intervals . Then J is open in I and nonempty . All we need then is to show that J is closed , for then it would follow that J = I . Suppose that t is a limit point of J in I

. Just as we argued in the first paragraph , there exists a curve y: ( t — e , t + e ) ^ $R^n$ such that y' ( t ) = F ( y ( t ) ) and y' ( t ) = 0 . Thus we may assume that y' = 0 on ( t — e , t + e ) , after replacing e by a smaller number . In particular y' ( t ) = 0 for some t G ( t — e , t + e ) n J , and there exists a matrix B such that B ■ y' ( t ) = x' ( t ) .

Now let y ( t ) := B ■ y ( t ) . Since F ( y ( t ) ) = y' ( t ) , we have F ( y ( t ) ) = y' ( t ) . Further , by construction y ( t ) = x ( t ) , so by uniqueness part of the previous result we must have y = x on ( t — e , t + e ) n J . Thus x is defined on J U ( t — e , t + e ) . But J was assumed to be maximal . So ( t — e , t + e ) C J . In particular t G J , which completes the proof that J is closed in I .

Theorem . Let X: I ^ M be a $C^1$ immersion . For every $t_0$ G I and $X_0$ G $T_Y$ ( $_{t0}$ ) M , there exists a unique parallel vector field X gX ( y ) such that X ( $t_0$ ) = $X_0$ .

Proof . First suppose that there exists a local chart ( U , 0 ) such that y: I ^ U is an embedding . Let X be a vector field on U and set X ( t ) := X ( y ( t ) ) . By ( 1 ) ,

Dy ( X ) ( t ) = $V_Y$ ( $_t$ ) X = <= ( y' ( t ) ( $X^{fc}$ ) + Y , $y^l$ ( t ) $X^j$ ( t ) rkj ( y ( t ) ) ) $E_{fc}$ ( y ( t ) ) .

fc ij

Further note that

y' ( t ) X = ( X o y ) ' ( t ) = X' ( t ) .

So , in order for X to be parallel along y we need to have

$X^k$ + Y $7^i$ ( i ) r{j - ( y ( i ) ) $X^j$ ( t ) = 0 ,

ij

for k = 1 , . . . , n . This is a linear system of ODE's in terms of $X^i$ , and therefore by the previous lemma it has a unique solution on I satisfying the initial conditions X $^i$ ( to ) = X0 .

Now let J C I be a compact interval which contains to - There exists a finite number of local charts of M which cover y ( J ) . Consequently there exist subintervals J\ , . . . , $J_n$ of J such that y embeds each $J_j$ into a local chart of M . Suppose that $t_0$ ssss $J_e$ , then , by the previous paragraph , we may extend $X_0$ to a parallel vector field defined on Je . Take an element of this extension which lies in a s u b i n t e r v a l J e intersecting $J_e$ and apply the previous paragraph to $J_e$ . Repeating this procedure , we obtain a parallel vector field on each Jj . By the uniqueness of each local extension mentioned above , these vector fields coincide on the overlaps of Jj . Thus we obtain a well - defined vector filed X on J which is a parallel extension of $X_0$ . Note that if J is any other compact subinterval of I which contains $t_0$ , and X is the parallel extension of $X_0$ on J , then X and X coincide on J n J , by the uniqueness of local parallel extensions . Thus , since each point of I is contained in a compact subinterval containing $t_0$ , we may consistently define X on all of I .

Finally let X be another parallel extension of $X_0$ defined on I . Let A C I be the set of points where X = X . Then A is closed , by continuity of X and X . Further A is open by the uniqueness of local extensions . Furthermore , A is nonempty since $t_0$ <= A . So A = I and we conclude that X is unique .

Using the previous result we now define , for every $X_0$ <= $T_Y$ ( $_{to}$ ) M ,

PW ( X0 ) := X ( t )

as the parallel transport of $X_0$ along 7 to $T_Y$ ( $_t$ ) M . Thus we obtain a mapping from

$^T$Y ( to ) $^{M\ to\ T}$Y ( t ) $^M$ .

# 14.4 GAUSS MAP

In differential geometry , the **Gauss map** ( named after Carl F . Gauss ) maps a surface in Euclidean space $\mathbf{R}^3$ to the unit sphere $S^2$ . Namely , given a surface X lying in $\mathbf{R}^3$ , the Gauss map is a continuous map $N: X \to S^2$ such that N ( p ) is a unit vector orthogonal to X at p , namely the normal vector to X at p .

The Gauss map can be defined ( globally ) if and only if the surface is orientable , in which case its degree is half the Euler characteristic .

The Gauss map can always be defined locally ( i . e . on a small piece of the surface ) . The Jacobian determinant of the Gauss map is equal to Gaussian curvature , and the differential of the Gauss map is called the shape operator .

Gauss first wrote a draft on the topic in 1825 and published in 1827 .

There is also a Gauss map for a link , which computes linking number .

The Gauss map can be defined for hypersurfaces in $\mathbf{R}^n$ as a map from a hypersurface to the unit sphere $S^{n-1} \subseteq \mathbf{R}^n$ .

For a general oriented k - submanifold of $\mathbf{R}^n$ the Gauss map can also be defined , and its target space is the oriented Grassmannian $G_{k,n}$ , i . e . the set of all oriented k - planes in $\mathbf{R}^n$ . In this case a point on the submanifold is mapped to its oriented tangent subspace . One can also map to its oriented normal subspace; these are equivalent as $G_{k,n} = G_{n-k,n}$ via orthogonal complement . In Euclidean 3 - space , this says that an oriented 2 - plane is characterized by an oriented 1 - line , equivalently a unit normal vector ( as $G_{1,n} = S^{n-1}$ ) , hence this is consistent with the definition above .

Finally , the notion of Gauss map can be generalized to an oriented submanifold X of dimension k in an oriented ambient Riemannian manifold M of dimension n . In that case , the Gauss map then goes from X to the set of tangent k - planes in the tangent bundle TM . The target space for the Gauss map N is a Grassmann bundle built on the tangent bundle TM . In the case where $M=R^n$ the tangent bundle is trivialized ( so the Grassmann bundle becomes a map to the Grassmannian ) , and we recover the previous definition .

## 14.5 HELIX

A helix plural helixes or helices is a type of smooth space curve , i . e . a curve in three dimensional space . It has the property that the tangent line at any point makes a constant angle with a fixed line called the axis . Examples of helices are coil springs and the handrails of spiral staircases

. A "filled - in" helix – for example , a "spiral" ( helical ) ramp – is called a helicoid . Helices are important in biology , as the DNA molecule is formed as two intertwined helices , and many proteins have helical substructures , known as alpha helices . The word helix comes from the Greek word twisted , curved

Helices can be either right - handed or left - handed . With the line of sight along the helix's axis , if a clockwise screwing motion moves the helix away from the observer , then it is called a right - handed helix; if towards the observer , then it is a left - handed helix . Handedness ( or chirality ) is a property of the helix , not of the perspective: a right handed helix cannot be turned to look like a left - handed one unless it is viewed in a mirror , and vice versa .

Most hardware screw threads are right - handed helices . The alpha helix in biology as well as the A and B forms of DNA are also right - handed helices . The Z form of DNA is left Handed .

The pitch of a helix is the height of one complete helix turn , measured parallel to the axis of the helix .

A double helix consists of two ( typically congruent ) helices with the same axis , differing by a translation along the axis .

A conic helix may be defined as a spiral on a conic surface , with the distance to the apex an exponential function of the angle indicating direction from the axis . An example is the Corkscrew roller coaster at Cedar Point amusement park .

A circular helix , ( i . e . one with constant radius ) has constant band curvature and constant torsion .

A curve is called a general helix or cylindrical helix[4] if its tangent makes a constant angle with a fixed line in space . A curve is a general helix if and only if the ratio of curvature to torsion is constant . [5]

Geometric pitch is the distance an element of an airplane propeller would advance in one revolution if it were moving along a helix having an angle equal to that between the chord of the element and a plane perpendicular to the propeller axis .

A curve is called a slant helix if its principal normal makes a constant angle with a fixed line in space . It can be constructed by applying a transformation to the moving frame of a general helix .

Some curves found in nature consist of multiple helices of different handedness joined together by transitions known as tendril perversions .

In mathematics , a helix is a curve in 3 - dimensional space . The following parametrisation in Cartesian coordinates defines a particular helix , Perhaps the simplest equations for one is

As the parameter t increases , the point ( x ( t ) , y ( t ) , z ( t ) ) traces a right - handed helix of pitch 2π ( or slope 1 ) and radius 1 about the z - axis , in a right - handed coordinate system .

In cylindrical coordinates ( r , θ , h ) , the same helix is parametrised by A circular helix of radius a and slope b / a ( or pitch 2πb ) is described by the following parametrisation: Another way of mathematically constructing a helix is to plot the complex - valued function $e^{xi}$ as a function of the real number x ( see Euler's formula ) . The value of x and the real and imaginary parts of the function value give this plot three real dimensions . Except for rotations , translations , and changes of scale , all right - handed helices are equivalent to the helix defined above . The equivalent left - handed helix can be constructed in a number of ways , the simplest being to negate any one of the x , y or z components .

Maps and Functions

The notation

f : X ^ Y

means that f is a function which assigns to every point x in the set X a point f ( x ) in the set Y . When Y = r we express this by saying that f is a real valued function defined on the set X and if Y is a vector space we may say that f is a vector valued function . However in general it is better to say that f is a map from X to Y and call the set X the source of the map and the set Y its target . The graph of f is the set

graph ( f ) := { ( x , y ) e X x Y | y = f ( x ) } .

Notes

We always distinguish two maps with the same graph when their targets are different .

A map f : X ^ Y is said to be

injective "| ( f ( xi ) = f ( x2 ) =^ xi = x2

surjective > iff < Vy e Y3x e X s . t . y = f ( x ) bijective

J [ it is both injective and surjective

.

Then

(a) f is injective it has a left inverse g : Y ^ X ( i . e . g o f = $id_X$ ) ;

(b) f is surjective it has a right inverse g : Y ^ X ( i . e . f o g = $id_Y$ ) ;

(c) f is bijective it has a two sided inverse $f^{-1}$ : Y ^ X .

( Item ( b ) is the Axiom of Choice . )

The analogous principle holds for linear maps: if A e $r^{mxn}$ then the linear map $r^n$ ^ $r^m$ : x ^ Ax is

(a) injective BA = $1_n$ for some B e $r^{nxm}$;

(b) surjective AB = $1_m$ for some B e $r^{nxm}$;

(c) bijective A is invertible ( i . e . m = n and det ( A ) = 0 ) .

( Here 1& is the k x k identity matrix . ) However , this principle fails com - pletely for continuous maps: the map f : [0 , 2n ) ^ $S^1$ defined by f ( 0 ) = ( cos 0 , sin 0 ) is continuous and bijective but its inverse is not continuous . ( Here $S^1$ C $r^2$ is the unit circle $x^2 + y^2 = 1$ . )

## 14.6 NORMAL FORMS

The Fundamental Idea of Differential Calculus is that near a point xo e U a smooth map f : U ^ V behaves like its linear approximation , i . e .

f ( x ) f ( x0 ) + df ( x0 ) ( x - x0 ) .

The Normal Form Theorem from Linear Algebra says that if A e $r^{mxn}$ has rank r then there are invertible matrices P e $R^{mxm}$ and Q e $r^{nxn}$ such that

$$P^{-1}AQ - \begin{pmatrix} 1\!\!1_r & 0_{r\times(n-r)} \\ 0_{(m-r)\times r} & 0_{(m-r)\times(n-r)} \end{pmatrix}.$$

By the Fundamental Idea we can expect an analogous theorem for smooth maps .

Theorem ( Local Normal Form for Smooth Maps ) . Let U C $r^n$ and V C $r^m$ be open , $x_0$ e U , and f : U ^ V be smooth . Assume that the derivative df ( $x_0$ ) e $R^{mxn}$ has rank r . Then there is an open neighborhood $U_0$ of $x_0$ in U , an open neighborhood $V_0$ of f ( $x_0$ ) in V , a diffeomorphism f : Ui x U2 C $r^r$ x $r^{n-r}$ , a diffeomorphism $ : V0 ^ Ui x V2 C $r^r$ x $r^{m-r}$ , such that f ( $x_0$ ) = ( 0 , 0 ) , $ ( f ( $x_0$ ) ) = ( 0 , 0 ) , and

□ $^1$ o f o f ( x , y ) = ( x , g ( x , y ) ) and dg ( 0 , 0 ) = 0

for ( x , y ) e Ui x U2 .

The Local Normal Form Theorem is an easy consequence of the Inverse Function Theorem .

( Inverse Function Theorem ) . Let U C $r^n$ , V C $r^m$ ,

$x_0$ E U and f : U ^ V be a smooth map . If df ( $x_0$ ) is invertible , then ( m = n and ) there are neighborhoods $U_0$ of $x_0$ in U and $V_0$ of f ( $x_0$ ) in V so that the restriction $f|_{U_0}$ : $U_0$ ^ $V_0$ is a diffeomorphism .

Here follow some other consequences of the Inverse Function Theorem .

Corollary ( Submersion Theorem ) . When r = m the diffeomorphisms f and f in Theorem A may be chosen so that the local normal form is

$f^{-1} \circ f \circ f ( x , y ) = x$ .

Corollary ( Immersion Theorem ) . When r = n the diffeomorphisms f and f in Theorem may be chosen so that the local normal form is

$f^{-1} \circ f \circ f ( x ) = ( x , 0 )$ .

Corollary ( Rank Theorem ) . If the rank of df ( x ) = r for all x E U

Notes

then for every $x_0$ E U the diffeomorphisms f and f in Theorem may be chosen so that the local normal form is

$f^{-1}$ o f o f ( x ) = ( x , 0 ) .

Corollary ( Implicit Function Theorem ) . Let U C $r^m$ x $r^n$ be an open set , let F : U ^ $r^n$ be smooth , and let ( $x_0$ , $y_0$ ) E U with $x_0$ E $r^m$ and $y_0$ E $r^n$ . Define the partial derivative $d_2F$ ( $x_0$ , $y_0$ ) E $r^{nxn}$ by

$d_2F$ ( xo , yo ) v := —t = 0

for v E $r^n$ . Assume that F ( $x_0$ , $y_0$ ) = 0 and that $d_2F$ ( $x_0$ , $y_0$ ) is invertible . Then there exist neighborhoods $U_0$ of $x_0$ in $r^m$ and $V_0$ of $y_0$ in $r^n$ and a smooth map g : $U_0$ ^ $V_0$ such that

Uo x V0 C U , g ( x0 ) = y0

and

F ( x , y ) = 0 y = g ( x ) for x E $U_0$ and y E $V_0$ .

Flat Spaces

Our aim in the next few sections is to give applications of the Cartan - Ambrose - Hicks Theorem . It is clear that the hypothesis T*R = R' for all developments will be difficult to verify without drastic hypotheses on the curvature . The most drastic such hypothesis is that the curvature vanishes identically .

Definition . A Riemannian manifold M is called flat if the Riemann curvature tensor R vanishes identically .

Theorem . Let M C $r^n$ be a smooth m - manifold .

(i) M is flat if and only if every point has a neighborhood which is isometric to an open subset of $r^m$ , i . e . at each point p e M there exist local coordinates $x^1$ , . . . , $x^m$ such that the coordinate vectorfields Ei = d / $dx^i$ are orthonormal .

(ii) Assume M is connected , simply connected , and complete . Then M is flat if and only if there is an isometry □ : M ^ $r^m$ onto Euclidean space .

Exercise . For b > a > 0 and c > 0 define M C $c^3$ by

$M := \{ ( u , v , w ) \in c^3 \setminus |u| = a , |v| = b , w = c u v \}$ .

Then M is diffeomorphic to a torus ( a product of two circles ) and M is flat . If $M'$ is similarly defined from numbers $b' > a' > 0$ and $c' > 0$ then there is an isometry $\$ : M \wedge M'$ if and only if ( a , b , c ) = ( a' , b' , c' ) , i . e . M = M' . ( Hint: Each circle $u = u_0$ is a geodesic as well as each circle v = vo; the numbers a , b , c can be computed from the length of the circle $u = u_0$ , the length of the circle $v = v_0$ , and the angle between them . )

Exercise ( Developable manifolds ) . Let n = m + 1 and let E ( t ) be a one - parameter family of hyperplanes in $r^n$ . Then there exists a smooth map $u : r \wedge r^n$ such that

$$E ( t ) = u , ( t )^{\pm} , |u ( t ) | = 1 ,$$

for every t . We assume that U ( t ) = 0 for every t so that u ( t ) and U ( t ) are linearly independent . Show that

$$L ( t ) := u , ( t )^{\pm} n U ( t )^{\pm} = \lim_{s \wedge t} E ( t ) n E ( s ) .$$

Thus L ( t ) is a linear subspace of dimension m — 1 . Now let $7 : r \wedge r^n$ be a smooth map such that

$$( {}^{A}f^{(t),u(t)} ) = \circ ( {}^{A}f^{(t),u(t))} = 0$$

for all t . This means that 7 ( t ) G E ( t ) and Y ( t ) G L ( t ) ; thus E ( t ) is spanned by L ( t ) and Y ( t ) . For t G r and e > 0 define

$$L ( t )_{<=} := \{v G L ( t ) \mid |v| < e\} .$$

Let I c r be a bounded open interval such that the restriction of 7 to the closure of I is injective . Prove that , for e > 0 sufficiently small , the set

$$M_0 := U ( {}^{7(t)} + {}^{L(t)} \wedge )$$

is a smooth manifold of dimension m = n — 1 . A manifold which arises this way is called developable . Show that the tangent spaces of $M_0$ are the original subspaces E ( t ) , i . e .

$$T_pM_0 = E ( t ) \text{ for } p G 7 ( t ) + L ( t )_{<=} .$$

( One therefore calls $M_0$ the "envelope" of the hyperplanes 7 ( t ) + E ( t ) . ) Show that $M_0$ is flat ( hint: use GauB - Codazzi ) . If ( T , 7 , 7' ) is a development of $M_0$ along $r^m$ , show that the map 0 : $M_0$ ^ $r^m$ , defined by

$$0 ( 7 ( t ) + v ) := Y' ( t ) + T ( t ) v$$

for v G L ( t ) $_e$ , is an isometry onto an open set M0 c $r^m$ . Thus a development "unrolls" $M_0$ onto the Euclidean space $r^m$ . When n = 3 and m = 2 one can visuali M C $r^{m+1}$

and a curve Y : r ^ M we may form the osculating developable $M_0$ to M along y by taking

) := $^T$Y ( t ) $^M$ .

This developable has common affine tangent spaces with M along 7 as

$^T{}_7$ ( t ) $^M$0 = $^{E(t)}$= $^T{}_7$ ( t ) $^M$

for every t . This gives a nice interpretation of parallel transport: $M_0$ may be unrolled onto a hyperplane where parallel transport has an obvious meaning and the identification of the tangent spaces thereby defines parallel transport in M .

Exercise . Each of the following is a developable surface in $r^3$ .

(i) A cone on a plane curve r c H , i . e .

M = {tp + ( 1 - t ) q |1 > 0 , q e r}

where H c $r^3$ is an affine hyperplane , p e $r^3$\H , and r c H is a 1 - manifold .

(ii) A cylinder on a plane curve r , i . e .

M = {q + tv | q e r , t e r}

where H and r are as in ( i ) and v is a fixed vector not parallel to H . ( This is a cone with the cone point p at infinity . )

(iii) The tangent developable to a space curve 7 : r ^ $r^3$ , i . e .

M = {Y ( t ) + sY ( t ) | |t - to| < e , 0 < s < e} , where $^{/}y(t_0)$ and $Y(t_0)$ are linearly independent and e > 0 is sufficiently small .

(iv) The paper model of a Mobius strip

Symmetric Spaces

In the last section we applied the Cartan - Ambrose - Hicks Theorem in the flat case; the hypothesis $*R = R' was easy to verify since both sides vanish . To find more general situations where we can verify this hypothesis note that for any development ( $ , 7 , 7' ) satisfying the initial conditions 7 ( 0 ) = $p_0$ , Y' ( 0 ) = $p'_0$ , and $ ( 0 ) = $\$_0$ , we have

$ ( t ) = ( t , 0 ) $o$y ( 0 , t ) so that the

hypothesis $*R = R' is certainly implied by the three hypotheses

$\$Y^{(t,0)}*^{R}pa \quad {}^{R}Y(t)$

$^{(t,0)}*^{R}P0 = {}^{R}Y>(t)$

$^{(}\$0^{)}*^{R}p_0 = {}^{R}{}_{p}{}'_{0}$ .

The last hypothesis is a condition on the initial linear isomorphism

$\$0 : {}^{T}p_0 \; M \wedge Tp\,/_{o}\, M'$

while the former hypotheses are conditions on M and M' respectively , namely , that the Riemann curvature tensor is invariant by parallel trans - port . It is rather amazing that this condition is equivalent to a rather simple geometric condition as we now show .

Symmetric Spaces

Definition . A Riemannian manifold M is called symmetric about the point p e M if there is a ( necessarily unique ) isometry f : M ^ M satisfying

$$f(p) = P, \; df(p) = -\,id .$$

M is called a symmetric space if it is symmetric about each of its points . A Riemannian manifold M is called locally symmetric about the point p e M if , for r > 0 sufficiently small , there is an isometry

$f : U_r(p, M) \wedge U_r(p, M), \; U_r(p, M) := \{q \; e \; M \mid d(p, q) < r\}$ , satisfying M is called a locally symmetric space if it is locally symmetric about each of its points .

Remark if M is locally symmetric , the isometry f : $U_r$ ( p , M ) ^ $U_r$ ( p , M ) with f ( p ) = p and df ( p ) = - id exists whenever 0 < r < inj ( p ) .

Exercise . Every symmetric space is complete . Hint: If 7 : I ^ M is a geodesic and 0 : M ^ M is a symmetry about the point 7 ( to ) for $t_0$ <= I then

$$^{0(}Y^{(t}0 + {}^{t))} = 7^{(}\text{to}^{-}$$

$^{t)}$ for all t <= r with $t_0$ +1 , $t_0$ — t <= I .

Theorem . Let M c $r^n$ be an m - dimensional submanifold . Then the following are equivalent .

(i) M is locally symmetric .

(ii) The covariant derivative VR ( defined below ) vanishes identically , i . e .

( V R ) p ( vi , V2 ) w = 0 for all p <= M and v , $v_i$ , $v_2$ , w <= $T_pM$ .

(iii) The curvature tensor R is invariant under parallel transport , i . e .

$$\$_7^{(t,} s ) *^R{}_7 ( s ) = {}^RY ( t )$$

for every smooth curve 7 : r ^ M and all s , t <= r .

Corollary . Let M and M' be locally symmetric spaces and fix two points $p_0$ <= M and $p'_0$ <= M' , and let $\$_0$ : $T_{p0}$ M ^ $T_p$ M' be an orthogonal linear isomorphism . Let r > 0 be less than the injectivity radius of M at $p_0$ and the injectivity radius of M' at $p'_0$ . Then the following holds .

(i) There is an isometry 0 : $U_r$ ( $p_0$ , M ) ^ $U_r$ ( $p'_0$ , M' ) with 0 ( $p_0$ ) = p0 and d0 ( $p_0$ ) = $\$_0$ if and only if $\$_0$ intertwines R and R':

$$^{(}\$0^{)} *^R \mathbf{po} = {}^R{}_p\text{o} .$$

(ii) Assume M and M' are connected , simply connected , and complete . Then there is an isometry 0 : M ^ M' with 0 ( $p_0$ ) = $p'_0$ and d0 ( $p_0$ ) = $T_0$ if and only if $T_0$

Proof . In ( i ) and ( ii ) the "only if" statement follows from Theorem ( Theorem Egregium ) with $T_0$ := d0 ( $p_0$ ) . To prove the "if" statement , let ( $ , 7 , 7' ) be a development satisfying 7 ( 0 ) = $p_0$ , Y' ( 0 ) = $p'_0$ , and T ( 0 ) = $T_0$ . Since R and R' are invariant under parallel transport , by Theorem it follows from the discussion in the beginning of this section that T*R = R' .

Corollary . A connected , simply connected , complete , locally symmetric space is symmetric .

Proof . Corollary with M' = M , $p'_0$ = $p_0$ , and $T_0$ = - id .

Corollary . A connected symmetric space M is homogeneous; i . e . given p , q e M there exists an isometry f : M ^ M with f ( p ) = q .

Proof . If M is simply connected the assertion follows from part ( ii ) of Corollary with M = M' , $p_0$ = p , p0 = q , and $T_0$ = $$_7$ ( 1 , 0 ) : $T_p$M ^ $T_q$M , where 7 : [0 , 1] ^ M is a curve from p to q . If M is not simply connected we can argue as follows . There is an equivalence relation on M defined by

p ~ q : 3 isometry f : M ^ M 7 f ( p ) = q .

Let p , q e M and suppose that d ( p , q ) < inj ( p ) . By Theorem there is a unique shortest geodesic 7 : [0 , 1] ^ M connecting p to q . Since M is symmetric there is an isometry f : M ^ M such that f ( q ( 1 / 2 ) ) = Y ( 1 / 2 ) and df ( Y ( 1 / 2 ) ) = - id . This isometry satisfies f ( q ( t ) ) = 7 ( 1 — t ) and hence f ( p ) = q . Thus p ~ q whenever d ( p , q ) < inj ( p ) . This shows that each equivalence class is open , hence each equivalence class is also closed , and hence there is only one equivalence class because M is connected .

**Check your Progress 1**

Discuss The e Frenet - Serret formula

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Notes

Discuss Normal Forms

## 14.7 LET US SUM UP

In this unit we have discussed the definition and example of The Frenet - Serret Formulas , Parallel Translation , Gauss Map , Helix , Normal Forms .

## 14.8 KEYWORDS

The Frenet - Serret Formulas ….. consider smooth curves given by a parametric equation in a three - dimensional space . That is , writing bold - face letters of vectors in three dimension

Parallel Translation ….. Let M be a manifold with a connection , and 7: I ^ M be an immersed curve .

Gauss Map ….. the **Gauss map** ( named after Carl F . Gauss ) maps a surface in Euclidean space

Helix ….. A helix plural helixes or helices is a type of smooth space curve , i . e . a curve in three - dimensional space

Normal Forms….. The Fundamental Idea of Differential Calculus is that near a point xo e U a smooth map f : U ^ V behaves like its linear approximation

## 14.9 QUESTIONS FOR REVIEW

Explain The Frenet - Serret Formulas , Normal Forms

## 14.10 SUGGESTED READINGS

Differential Geometry, Differential Geometry & Application, Introduction to Defferential Geometry, Basic of Differential Geometry.

## 14.11 ANSWERS TO CHECK YOUR PROGRESS

The Frenet - Serret Formulas , Normal Forms

( answer for Check your Progress - 1 Q )